Completeness

Th (Cantor)

X complete MS

$F_n \subset X \neq \phi$, closed $(n \in \mathbb{N})$

$F_1 \supset F_2 \supset \cdots$

$\delta(F_n) = \sup\{d(x,y) \mid x, y \in F_n\} \to 0$

$\Longrightarrow \exists 1\, x^* \in \bigcap_{n=1}^{\infty} F_n$

Proof.

⟨Existence⟩

As $F_n \neq \phi$, let $x_n \in F_n$ $(n \in \mathbb{N})$.

$\{x_n\} \subset X$: Cauchy sequence

Let $m, n \in \mathbb{N}$: $m \geqq n$.

As $F_n \supset F_m$, we have $x_m, x_n \in F_n$.

Thus, $d(x_m, x_n) \leqq \delta(F_n) \to 0$

as $m, n \to \infty$

This shows that

$\{x_n\}$ is a Cauchy sequence. ⌟

As X is complete, $\exists x^* \in X : x_n \to x^*$.

$$x^* \in \bigcap_{n=1}^{\infty} F_n$$

i.e. $\forall n \in \mathbb{N}$, $x^* \in F_n$.

Let $n \in \mathbb{N}$ and fix it.

As $F_n \supset F_{n+1} \supset \cdots$,

$\{x_n, x_{n+1}, x_{n+2}, \cdots\} \subset F_n$.

As $\{x_m\}_{m \geq n} \subset F_n$, $x_m \to x^*$,

and F_n is closed in X,

we have that $x^* \in F_n$ $(\forall n \in \mathbb{N})$.

$\therefore x^* \in \bigcap_{n=1}^{\infty} F_n$. $\lrcorner$

<Uniqueness>

Let $x, y \in \bigcap_{n=1}^{\infty} F_n$.

We show that $\underline{x = y}$.

As $x, y \in F_n$ $(n \in \mathbb{N})$,

$d(x, y) \leq \delta(F_n)$ $(\forall n \in \mathbb{N})$.

As $\delta(F_n) \to 0$, we obtain $d(x, y) = 0$.

$\therefore x = y$.

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ex (without completeness)

$X = (0, 2)$ not complete

$H_n = \left(0, \frac{1}{n}\right]$ $(n \in \mathbb{N})$

Then, $H_n \subset X$ $\neq \phi$, closed in X

$H_1 \supset H_2 \supset \cdots$

$\delta(H_n) \to 0$

However, $\bigcap_{n=1}^{\infty} H_n = \phi$.

ex (without closedness)

$X = [0, 1]$ complete

$G_n = \left(0, \frac{1}{n}\right)$ $(n \in \mathbb{N})$

Then, $G_n \subset X$ $\neq \phi$, not closed in X

$G_1 \supset G_2 \supset \cdots$

$\delta(G_n) \to 0$

However, $\bigcap_{n=1}^{\infty} G_n = \phi$.

ex (without $\delta(F_n) \to 0$)

$X = \mathbb{R}$ complete

$F_n = [n, \infty) \quad (n \in \mathbb{N})$

Then, $F_n \neq \phi$, closed in X.

$F_1 \supset F_2 \supset \cdots\cdots$

$\delta(F_n) \not\to 0$

$(\forall n \in \mathbb{N},\ \delta(F_n) = \infty)$

In this case, $\bigcap_{n=1}^{\infty} F_n = \phi$

Def.

X, Y MSs

$f : X \to Y$ contraction mapping

$\iff \exists r \in (0,1) : \forall x, y \in X,$

$$d(f(x), f(y)) \leq r\, d(x,y)$$

$\iff \exists r \in [0,1) : \forall x, y \in X,$

$$d(f(x), f(y)) \leq r\, d(x,y)$$

※ f is also called

a r-contraction mapping.

X, Y MSs

$f : X \to Y$ contraction

$\Rightarrow f$: continuous

Proof

Let $x \in X$ and $(x_n) \subset X : x_n \to x$.

We show that $f(x_n) \to f(x)$.

i.e. $d(f(x_n), f(x)) \to 0$

As f is a contraction mapping,

$\exists r \in (0,1) : \forall u, v \in X$,

$$d(f(u), f(v)) \leq r\, d(u, v).$$

Thus, for all $n \in \mathbb{N}$,

$$0 \leq d(f(x_n), f(x)) \leq \underbrace{r\, d(x_n, x)}_{\to 0}$$

Consequently, we obtain

$$f(x_n) \to f(x).$$

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ex

$a \in (-1, 1)$

$f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = ax \quad {}^{\forall} x \in \mathbb{R}$$

$\Rightarrow f : |a|$-contraction

i.e. $|f(x) - f(y)| \leqq |a| |x - y|$

$${}^{\forall} x, y \in \mathbb{R}$$

Proof

Let $x, y \in \mathbb{R}$.

It holds true that

$$\begin{aligned} &|f(x) - f(y)| \\ &= |ax - ay| \\ &= |a| |x - y| \\ &\leqq |a| |x - y|. \end{aligned}$$

//

ex

$a \in (-1, 1)$

$f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = a|x| \quad \forall x \in \mathbb{R}$$

$\Rightarrow f : |a|$-contraction

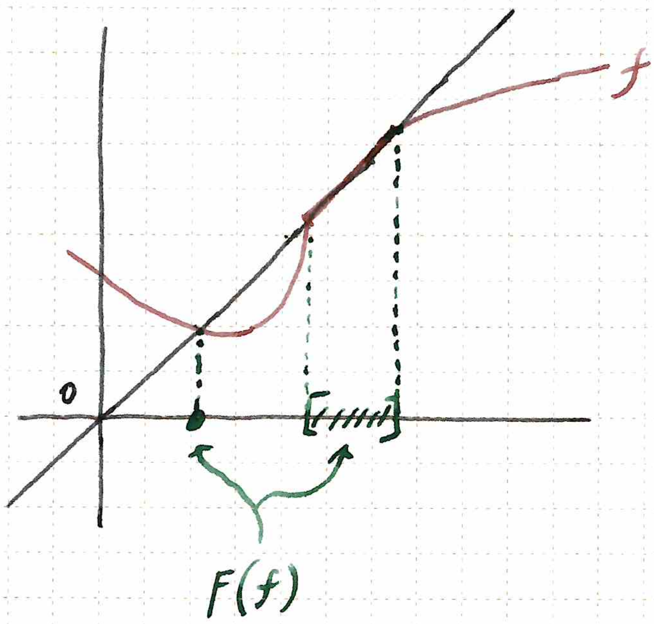
Def

$X \neq \phi$

$f : X \to X$

$F(f) = \{ u \in X \mid f(u) = u \}$

the set of <u>fixed points</u> of f



Remark

X MS

$f : X \to X$

Then,

$x \in F(f)$

$\Longleftrightarrow f(x) = x$

$\Longleftrightarrow d(x, f(x)) = 0$

$\Longleftrightarrow d(x, f(x)) \leqq 0$

X MS

(d1) $d(x, y) \geqq 0$; $d(x, y) = 0 \Longleftrightarrow x = y$

(d2) $d(x, y) = d(y, x)$

(d3) $d(x, y) \leqq d(x, z) + d(z, y)$

X MS

$f: X \to X$ r-contraction

i.e. $\exists r \in (0,1): \forall x, y \in X,$

$$d(f(x), f(y)) \leq r\, d(x, y)$$

$x^*, y^* \in F(f)$

$\Rightarrow x^* = y^*$

Proof

Using $x^*, y^* \in F(f)$, we have

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq r\, d(x^*, y^*). \quad -(*)$$

If $x^* \neq y^*$, then $d(x^*, y^*) > 0$.

Therefore, dividing the both sides of $(*)$ by $d(x^*, y^*)$ (>0), we obtain

$$1 \leq r.$$

This is a contradiction.

$\therefore x^* = y^*$.

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X MS

$f: X \to X$ continuous

$x \in X$

$x_n = f^n(x) \quad (n \in \mathbb{N})$

$x_n \to x^*$ — $(*)$

$\Longrightarrow x^* \in F(f)$

Proof

We prove that $\underline{x^* \in F(f)}$.

i.e. $f(x^*) = x^*$

As $x_n = f^n(x)$,

$$x_1 = f(x)$$

$$x_2 = f^2(x) = f\big(f(x)\big) = f(x_1)$$

$$x_3 = f^3(x) = f\big(f^2(x)\big) = f(x_2)$$

$$\cdots$$

$$x_n = f(x_{n-1}).$$

Thus, as f is continuous,

$$f(x^*) \overset{(*)}{=} f(\lim x_n)$$

$$= \lim f(x_n)$$

$$= \lim x_{n+1}$$

$$= x^*$$

f: continuous

$(*)$

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Th

X complete M.S

$f: X \to X$ r-contraction

$\Rightarrow \exists 1\, x^* \in F(f)$:

$\forall x \in X,\ f^n(x) \to x^* \ (n \to \infty)$

Proof

Let $x \in X$, and

define $x_n = f^n(x) \quad (n \in \mathbb{N} \cup \{0\})$.

(Then, $x_0 = x$, $x_1 = f(x)$,

$x_2 = f(x_1) = f^2(x)$, ---)

$\{x_n\} \subset X$: Cauchy sequence.

It holds that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f^n(x), f^{n+1}(x)) \\ &\leqq r\, d(f^{n-1}(x), f^n(x)) \\ &\leqq r^2 d(f^{n-2}(x), f^{n-1}(x)) \\ &\leqq \cdots \\ &\leqq r^n d(x, f(x)) \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

(*)

Let $m, n \in \mathbb{N} : m \geqq n$.

Then,

$$
\begin{aligned}
& d(x_n, x_m) \\
& \leqq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots \\
& \qquad \cdots + d(x_{m-1}, x_m)
\end{aligned}
$$

Using (*), we have

$$
\begin{aligned}
& d(x_n, x_m) \\
& \quad \leqq r^n d(x, x_1) + r^{n+1} d(x, x_1) + \cdots \\
& \qquad \cdots + r^{m-1} d(x, x_1) \\
& \quad \leqq r^n d(x, x_1)(1 + r + r^2 + \cdots) \\
& \quad = \frac{r^n}{1-r} d(x, x_1) \\
& \quad \longrightarrow 0 \quad (\text{as } m, n \to \infty)
\end{aligned}
$$

This shows that $\{x_n\}$ is a Cauchy sequence. $\lrcorner$

As X is complete,

$\exists x^* \in X : x_n \longrightarrow x^*$.

$\underline{x^* \in F(f)}$ i.e. $x^* = f(x^*)$

It follows that

$$\left.\begin{aligned} f(x^*) &= f(\lim x_n) \\ &= \lim f(x_n) \\ &= \lim x_{n+1} \\ &= x^*. \end{aligned}\right\rfloor$$

f: continuous

<Uniqueness>

Let $u, v \in F(f)$. $\therefore u = f(u),\ v = f(v)$

Our aim is to prove that $\underline{u = v}$.

As f is a r-contraction,

$$\begin{aligned} d(\underline{f(u)}, \underline{f(v)}) &\leq r\, d(u,v) \\ \| \\ d(\underline{u}, \underline{v}) \end{aligned}$$

Thus, $(1-r)d(u,v) \leq 0$.

As $r < 1$, we obtain $d(u,v) \leq 0$.

$\therefore u = v$.

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Completeness

1. (Cantorの定理) Xを完備距離空間, F_n $(n \in \mathbb{N})$ をその空でない閉集合の列とする. また,

$$F_1 \supset F_2 \supset F_3 \supset \cdots \quad \text{and}$$
$$\delta(F_n) \equiv \sup\{d(x,y) : x,y \in F_n\} \to 0 \quad (n \to \infty)$$

を仮定する. このとき, $\bigcap_{n\in\mathbb{N}} F_n$はただ一つの要素からなることを示せ.

2. Cantorの定理において, (a) Xの完備性, (b) F_nの閉性, (c) $\delta(F_n) \to 0$ のそれぞれが満たされないために定理が成り立たなくなる例を挙げよ.

3. 縮小写像の定義を述べ, 例を挙げて説明せよ. また, 縮小写像は連続であることを示せ.

4. 離散距離空間X上で定義された縮小写像$f : X \to X$はどのような写像か考えてみよ.

5. Xを距離空間, $f : X \to X$を縮小写像とする. このとき, fの不動点の一意性(fに不動点が存在するならば, それはただ一つであること)を証明せよ.

6. Xを距離空間, $f : X \to X$を連続な写像とする. Xの要素xに対して$x_n = f^n(x)$ $(n \in \mathbb{N} \cup \{0\})$ により点列$\{x_n\}$を定義する. 点列$\{x_n\}$が$x^* \in X$に収束したとすると, x^*はfの不動点定理である. このことを示せ.

7. 縮小写像の不動点定理を証明せよ.

8. 縮小写像の不動点定理の証明において, 空間の完備性, 縮小写像の条件, そして写像の連続性がどのように用いられているか確認せよ.