Lipschitz mappings

Thm (Rolle)

$f : [a,b] \to \mathbb{R}$ continuous,
differentiable on (a,b)

$f(a) = f(b)$

$\Rightarrow \exists c \in (a,b) : f'(c) = 0$

Proof

Assume, w.l.g., that
f is not a constant function.

As f is continuous on a compact set $[a,b]$,

$$\exists c \in [a,b] : f(c) = \sup_{x \in [a,b]} f(x) \ ;$$

$$\exists d \in [a,b] : f(d) = \inf_{x \in [a,b]} f(x) .$$

As f is not constant,

$$f(a) = f(b) < f(c) \text{ or}$$

$$f(d) < f(a) = f(b) .$$

Assume, w.l.g., that $f(a) = f(b) < f(c)$.

Clearly, $a < c < b$.

It holds that

$$f(x) \leq f(c) \quad \forall x \in [a, b]. \quad -(*)$$

As f is differentiable on (a, b),

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

From $(*)$,

$$f'(c) = \lim_{x \downarrow c} \frac{f(x) - f(c)}{x - c} \leq 0 ;$$

$$f'(c) = \lim_{x \uparrow c} \frac{f(x) - f(c)}{x - c} \geq 0 .$$

Hence, $f'(c) = 0$.

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$\underline{\underline{\text{Th}}}$ (the mean value theorem)

$f : [a,b] \to \mathbb{R}$ continuous,

differentiable on (a,b)

$$\Rightarrow {}^{\exists} c \in (a,b) : f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof

Let $k = \dfrac{f(b)-f(a)}{b-a} \in \mathbb{R}$.

Using this k, define $F : [a,b] \to \mathbb{R}$ as

$$F(x) = f(x) - f(a) - k(x-a) \quad {}^{\forall} x \in [a,b].$$

Then, F is $\begin{cases} \text{continuous on } [a,b] \\ \text{differentiable on } (a,b). \end{cases}$

Furthermore, $F(a) = F(b) = 0$.

From $\underline{\underline{\text{Th}}}$ (Rolle),

$${}^{\exists} c \in (a,b) : F'(c) = 0.$$

$$\therefore f'(c) = k = \frac{f(b)-f(a)}{b-a}.$$

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Def.

(X, d_X), (Y, d_Y) metric spaces

$f: X \to Y$ Lipschitz

$\Longleftrightarrow \exists K \geqq 0 : {}^\forall x, y \in X,$

$$d_Y(f(x), f(y)) \leqq K\, d_X(x, y)$$

※ f is called K-Lipschitz.

$\mathbb{R}$においては、

$f: \mathbb{R} \to \mathbb{R}$ Lipschitz

$\Longleftrightarrow \exists K \geqq 0 : {}^\forall x, y \in \mathbb{R},$

$$|f(x) - f(y)| \leqq K\, |x - y|.$$

$f: X \to Y$ 1-Lipschitz

$\Longleftrightarrow {}^{\forall} x, y \in X,\ d(f(x), f(y)) \leqq d(x, y)$

$\Longleftrightarrow f$: nonexpansive (非拡大)
(NE)

$\downarrow$

$\mathbb{R}$において

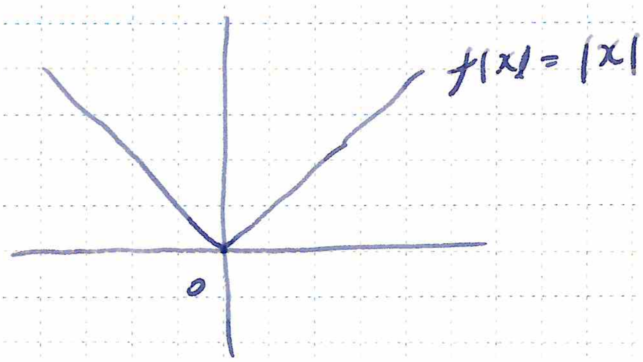
$f: \mathbb{R} \to \mathbb{R}$ NE

$\Longleftrightarrow {}^{\forall} x, y \in \mathbb{R},\ |f(x) - f(y)| \leqq |x - y|.$

<u>ex</u>

$f(x) = |x|$: NE

$\Longleftrightarrow {}^{\forall} x, y \in \mathbb{R},\ \big||x| - |y|\big| \leqq |x - y|$



X, Y MSs
$f: X \to Y$ Lipschitz
$\Rightarrow f$: continuous

<u>Proof.</u>

Let $x \in X$ and let $\{x_n\} \subset X : x_n \to x$.

We prove that $\underline{f(x_n) \to f(x)}$.

As f is a Lipschitz mapping,

$\exists K \geqq 0 : \forall n \in \mathbb{N}$,

$$d\big(f(x_n), f(x)\big) \leqq K\, d(x_n, x).$$

As $d(x_n, x) \to 0$, we obtain

$$d\big(f(x_n), f(x)\big) \to 0.$$

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Th

$I \subset \mathbb{R}$ open interval

$f : I \to \mathbb{R}$ differentiable

$\Rightarrow$ Equivalent

① f : K-Lipschitz

② $|f'(x)| \leq K \quad \forall x \in I$

Proof

① $\Rightarrow$ ②

Let $x, y \in I : x \neq y$.

From ①,

$$|f(x) - f(y)| \leq K |x - y|.$$

As $x \neq y$,

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq K.$$

Therefore, $\displaystyle \lim_{y \to x} \left| \frac{f(y) - f(x)}{y - x} \right| \leq K$

$$\therefore \left| \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \right| \leq K.$$

$$\therefore |f'(x)| \leq K \quad \forall x \in I. \quad \lrcorner$$

② $\Rightarrow$ ①

Let $x, y \in I$.

Our goal is to show that

$$|f(x) - f(y)| \leq K |x - y|.$$

Assume, w.l.g., that $x < y$.

From the mean value theorem,

$$\exists c \in (x, y) : f'(c) = \frac{f(x) - f(y)}{x - y}.$$

$$\therefore |f(x) - f(y)| = |f'(c)| \, |x - y| \leq K |x - y|. \quad ②$$

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ex

$f(x) = K \sin x$

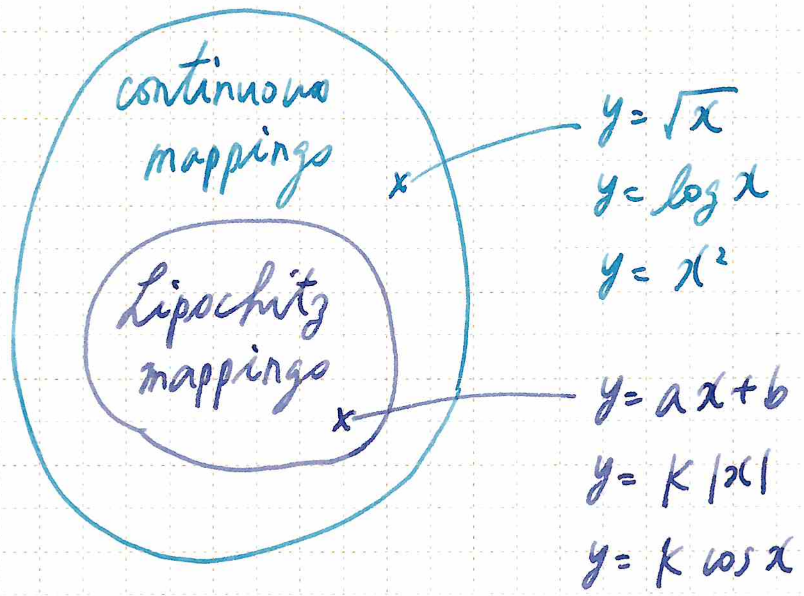
$\Rightarrow f$: K-Lipschitz

ex

$f_1(x) = \sqrt{x}$

$f_2(x) = \log x$

$\Rightarrow f_1, f_2$ are not Lipschitz mappings.



X MS

$f : X \to \mathbb{R}$ K-Lipschitz

$\Rightarrow |f|$: K-Lipschitz

Proof

Let $x, y \in X$.

We show that

$$\big|\,|f|(x) - |f|(y)\,\big| \leq K\, d(x,y).$$

As f is K-Lipschitz,

$$|f(x) - f(y)| \leq K\, d(x, y). \quad \text{— } (*)$$

It holds that

$$\begin{aligned} &\big|\,|f|(x) - |f|(y)\,\big| \\ &= \big|\,|f(x)| - |f(y)|\,\big| \\ &\leq |f(x) - f(y)| \\ &\leq K\, d(x, y). \quad (*) \end{aligned}$$

This completes the proof.

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X MS

$f : X \to \mathbb{R}$ K_1-Lipschitz

$g : X \to \mathbb{R}$ K_2-Lipschitz

$\alpha, \beta \in \mathbb{R}$

$\Rightarrow \alpha f + \beta g$: $|\alpha| K_1 + |\beta| K_2$-Lipschitz

Proof

Let $x, y \in X$.

We obtain

$$
\begin{aligned}
&|(\alpha f + \beta g)(x) - (\alpha f + \beta g)(y)| \\
&= |\alpha f(x) + \beta g(x) - \{\alpha f(y) + \beta g(y)\}| \\
&= |\alpha (f(x) - f(y)) + \beta (g(x) - g(y))| \\
&\leq |\alpha| \, |f(x) - f(y)| + |\beta| \, |g(x) - g(y)| \\
&\leq |\alpha| \cdot K_1 \cdot d(x, y) + |\beta| \cdot K_2 \cdot d(x, y) \\
&= (|\alpha| K_1 + |\beta| K_2) \cdot d(x, y).
\end{aligned}
$$

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Remark

$f, g : X \to \mathbb{R}$ Lipschitz

$\not\Rightarrow fg, \ \frac{f}{g}$: Lipschitz

ex

$X = \mathbb{R}$

$f(x) = g(x) = x$ Lipschitz

However, $(fg)(x) = x^2$ is not a Lipschitz function.

ex

$X = [1, \infty)$

$\left.\begin{aligned} f(x) &= x \\ g(x) &= \frac{1}{x} \end{aligned}\right\}$ Lipschitz

However, $\left(\frac{f}{g}\right)(x) = x^2$ is not a Lipschitz function.

$$\boxed{\begin{array}{l} X, Y \text{ MSs} \\ f: X \to Y \text{ K-Lipschitz} \\ A \subset X \text{ bdd} \\ \Rightarrow f(A) \, (\subset Y) : \text{bdd} \end{array}}$$

<u>Proof</u>

As A is bdd,

$\exists M > 0 : \forall x, y \in A, \ d(x, y) \leq M$. — (*)

Define $M' \equiv KM > 0$.

Let $u, v \in f(A)$.

Then, $\exists x, y \in A : \begin{cases} u = f(x) \\ v = f(y) \end{cases}$.

It holds that

$$\begin{aligned} d(u, v) &= d(f(x), f(y)) \quad \big\} \ f: \text{K-Lipschitz} \\ &\leq K \, d(x, y) \quad \big\} \ (*) \\ &\leq KM \\ &= M'. \end{aligned}$$

$\therefore \exists M' > 0 : \forall u, v \in f(A), \ d(u, v) \leq M'$.

This shows that $f(A) \, (\subset Y)$ is bdd. //

Cor

X, Y MSs

$f: X \to Y$ Lipschitz

$\{x_n\} \subset X$ bdd

$\Rightarrow \{f(x_n)\} (\subset Y)$: bdd

Lipschitz mappings

1. 最大値最小値の定理を用いてロルの定理を証明せよ。

2. 平均値の定理を証明せよ。

3. Lipschitz写像の定義とある写像がLipschitz写像ではないこと正確に述べよ。

4. $f : I \to \mathbb{R}$とする。ここで、Iは$\mathbb{R}$の区間である。fは連続で、Iの内部では微分可能とする。このとき、fがK-Lipschitz関数であることは、

$$|f(x)| \le K \quad \forall x \in I$$

と同値である。これを示せ。

5. Lipschitz写像は連続である. このことを証明せよ.

6. 連続だがLipschitz写像ではない例を2つ挙げよ。

7. Xを距離空間, fをX上の実数値, K-Lipschitz関数とする. このとき, $|f|$もK-Lipschitz関数であることを示せ.

8. Xを距離空間, $f : X \to \mathbb{R}$をK_1-Lipschitz関数, $g : X \to \mathbb{R}$をK_2-Lipschitz関数とする. また, $a, b \in \mathbb{R}$とする. このとき, $af + bg$は$(|a|K_1 + |b|K_2)$-Lipschitz関数であることを示せ.

9. X, Y, Zを距離空間, $f : X \to Y$をK_1-Lipschitz関数, $g : Y \to Z$をK_2-Lipschitz関数とする. このとき, 合成関数$g \circ f : X \to Z$はK_1K_2-Lipschitz関数になる. これを証明せよ.

10. X, Yを距離空間, AをXの有界集合, $f : X \to Y$をK-Lipschitz関数とする. このとき, $f(A)$はYの有界集合であることを示せ.

11. Xを距離空間, $f : X \to \mathbb{R}$をK_1-Lipschitz関数, $g : X \to \mathbb{R}$をK_2-Lipschitz関数とする. このとき, fgはLipschitz関数にならないが、仮定を追加することでfgがLipschitz関数になるようにできないかを考え, 自分で定理を作ってみよ.