Compactness (2)

Compactness in metric spaces

X compact M.S
$A \subset X$: $\#A = \infty$
$\Rightarrow A^d \neq \emptyset$

Proof.

We show that $\exists x \in A^d$.

i.e. $\exists x \in X: \forall r > 0, S_r(x) \cap (A \setminus \{x\}) \neq \emptyset$

Suppose by way of contradiction that

$\forall x \in X, \exists r(x) > 0: S_{r(x)}(x) \cap (A \setminus \{x\}) = \emptyset$.

i.e $\forall x \in X, \exists r(x) > 0: S_{r(x)}(x) \cap A \cap \{x\}^c = \emptyset$

i.e. $\forall x \in X, \exists r(x) > 0: S_{r(x)}(x) \cap A \subset \{x\}$. $-(*)$

As $X = \bigcup_{x \in X} S_{r(x)}(x)$ and X is compact,

$\exists \{x_1, \cdots, x_N\} \subset X : X = \bigcup_{i=1}^{N} S_{r_i}(x_i)$, $-(**)$

where $r_i = r(x_i)$.

From $(*)$ and $(**)$,

$$
\begin{aligned}
\{x_1, \cdots, x_N\} &\supset \bigcup_{i=1}^{N} \left(S_{r_i}(x_i) \cap A\right) \\
&= \left(\bigcup_{i=1}^{N} S_{r_i}(x_i)\right) \cap A \\
&\overset{(**)}{=} X \cap A \\
&= A.
\end{aligned}
$$

This contradicts $\#A = \infty$.

//

Cor

X compact MS

$\{x_n\} \subset X$

$\Rightarrow \exists \{x_{n_i}\} \subset \{x_n\}, x \in X : x_{n_i} \to x$

Review

(X, d) MS

$A \subset X$

$\Rightarrow$ Equivalent

① $x \in A^d$

② $\exists \{x_n\} \subset A : \begin{cases} x_n \to x \\ x_n \neq x \end{cases}$

ex

$[0,1] \subset \mathbb{R}$ compact

$\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

$\Rightarrow \exists \{x_{n_i}\} = \{x_n\},\ 0 \in [0,1] : x_{n_i} \to 0.$

ex

$(0,1]\ (\subset \mathbb{R})$: not compact

$\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\} \subset (0,1]$

However,

$\forall \{x_{n_i}\} \subset \{x_n\},\ x \in (0,1] : x_{n_i} \not\to x.$

ex

$[0,1]\ (\subset \mathbb{R})$

with the discrete metric

$\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\} \subset [0,1]$

However,

$\forall \{x_{n_i}\} \subset \{x_n\},\ x \in [0,1],\ x_{n_i} \not\to x.$

In this case, $[0,1]$ is not compact.

X compact metric space
$\Rightarrow X$: complete

Proof

Let $\{x_n\} \subset X$ be a Cauchy seq.

As X is compact,

$\exists \{x_{n_i}\} \subset \{x_n\}, x \in X : x_{n_i} \to x$.

Then, $x_n \to x$.

This shows that $\{x_n\}$ is convergent, and X is complete.

//

Review

X MS
$\{x_n\} \subset X$ Cauchy seq.
$\exists \{x_{n_i}\} \subset \{x_n\}, x \in X : x_{n_i} \to x$
$\Rightarrow x_n \to x$

Cor

X metric space
$A \subset X$ compact
$\Rightarrow A$: closed in X.

Proof

As A is compact, it is complete.
Therefore, A is closed in X. //

Cor

X MS

$A \subset X$ compact

$\Rightarrow A$ is closed in X.

Alternative Proof

Let $\{x_n\} \subset A : x_n \to x \in X$.

We show that $x \in A$.

As $\{x_n\} \subset A$ and A is compact,

$\exists \{x_{n_i}\} \subset \{x_n\}$, $y \in A : x_{n_i} \to y$.

As $x_n \to x$, we have $x_{n_i} \to x$.

Thus, $x = y$.

It holds true that $x (= y) \in A$.

//

X MS
$A \subset X$ compact
$\Rightarrow A$: bdd
i.e. $\exists M \geq 0 : \forall x, y \in A,\ d(x,y) \leq M$

<u>Proof</u>

Suppose by way of contradiction that A is not bdd.

Then, $\forall M \geq 0,\ \exists x, y \in A : d(x,y) > M$.

$\therefore \forall n \in \mathbb{N} : \exists x_n, y_n \in A : d(x_n, y_n) > n$.

$\therefore \exists \{x_n\}, \{y_n\} \subset A : d(x_n, y_n) \to \infty$. — (*)

As $\{x_n\} \subset A$ and A is compact,

$\exists \{x_{n_i}\} \subset \{x_n\},\ x \in A : x_{n_i} \to x$.

As $\{y_{n_i}\} \subset \{y_n\} \subset A$ and A is compact,

$\exists \{y_{n_j}\} \subset \{y_{n_i}\},\ y \in A : y_{n_j} \to y$.

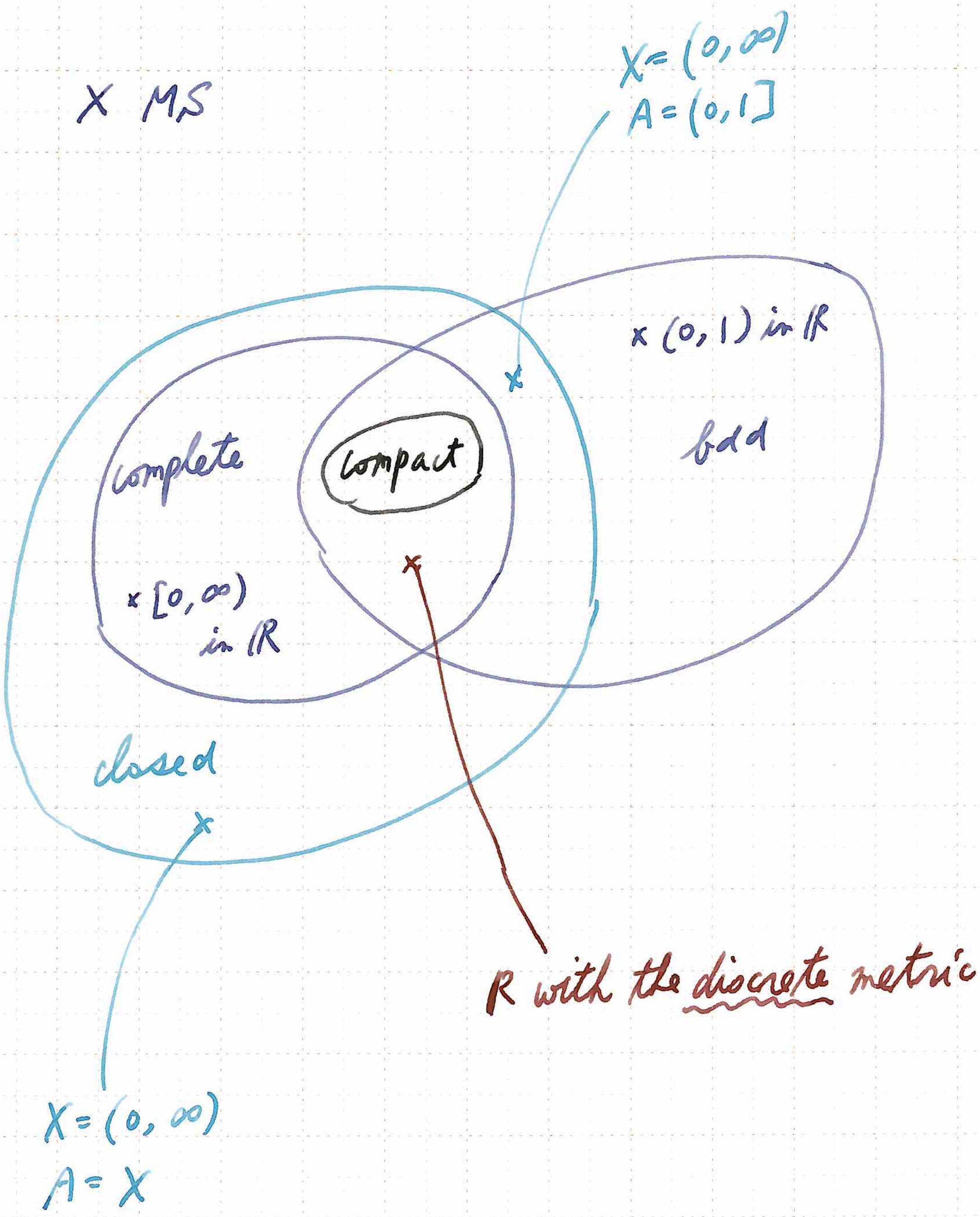
It holds that as $\{x_{n_j}\} \subset \{x_{n_i}\}$, $x_{n_j} \to x$.

Therefore, $d(x_{n_j}, y_{n_j}) \to d(x,y) \in \mathbb{R}$.

This contradicts (*).

//

X MS


X = (0, ∞)
A = (0, 1]
× (0, 1) in ℝ
complete
compact
bdd
× [0, ∞) in ℝ
closed
R with the discrete metric
X = (0, ∞)
A = X

Th

> X compact top. space
> $f: X \to \mathbb{R}$ continuous
> $\Rightarrow f$ attains maximum and minimum on X.

Proof

As X is compact and f is continuous, $f(X)$ $(\subset \mathbb{R})$ is also compact.

Consequently, $f(X)$ is closed and bdd in $\mathbb{R}$.

Thus, $^\exists \alpha = \max f(X)$ $(\in \mathbb{R})$.

$\therefore\ ^\exists x^* \in X : f(x^*) = \alpha = \max f(X)$.

Similarly, $^\exists \beta = \min f(X)$ $(\in \mathbb{R})$.

$\therefore\ ^\exists x_* \in X : f(x_*) = \beta = \min f(X)$.

//

Review

Th

$$\boxed{\begin{array}{l} (X, \mathbb{G}), (Y, \mathbb{H}) \text{ top. spaces} \\ A \subset X \text{ compact} \\ f : X \to Y \text{ continuous} \\ \Rightarrow f(A)\ (\subset Y) : \text{compact} \end{array}}$$

$$\boxed{\begin{array}{l} A \subset \mathbb{R} \neq \phi, \text{ bdd, closed} \\ \Rightarrow \exists \alpha = \max A \ (\in A) \\ \quad \exists \beta = \min A \ (\in A) \end{array}}$$

Cor

X compact top. space

$f, g : X \to \mathbb{R}$ continuous

$\alpha \in \mathbb{R}$

$\Rightarrow$ The following functions attain maximum and minimum on X:

(1) $|f|$, (2) $f+g$,

(3) $f \cdot g$, (4) αf

Proof

As $|f|$, $f+g$, fg, αf have X as their domain and they are continuous, we have the desired result. //

Remark

X compact top. space

$f, g : X \to \mathbb{R}$ continuous

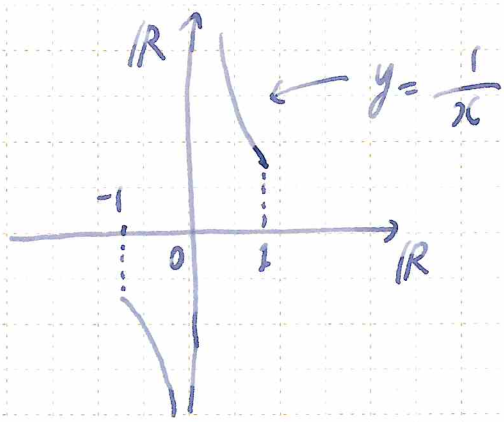
$\not\Rightarrow$ $\dfrac{f}{g}$ attains its maximum and minimum values on X.

ex

$X = [-1, 1]$ ($\subset \mathbb{R}$) compact

$f(x) = 1$, $g(x) = x$ continuous

However, $\dfrac{f}{g}(x) = \dfrac{1}{x}$ has no max or min values on $X = [-1, 1]$.



Review

$(X, \mathfrak{S})$ compact top. space
$A \subset X$ closed in X, $\neq \emptyset$
$\Rightarrow A$: compact

Cor

X compact MS
$F \subset X$, $\neq \emptyset$
$\Rightarrow$ Equivalent
① F : compact
② F : closed in X.

$F \subset \mathbb{R}$, $\neq \phi$
Then, F : compact
$\Leftrightarrow$ F : bdd, closed in $\mathbb{R}$.

<u>Proof</u>

($\Rightarrow$) OK.

($\Leftarrow$) As F is bdd,

$$\exists [a, b] \subset \mathbb{R} : F \subset [a, b].$$

As F is closed in $\mathbb{R}$, so is in $[a, b]$.

As $[a, b]$ is compact and
F is closed in $[a, b]$,
we have that F is compact.

//

<u>Review</u>

X MS
$C \subset X$ closed
$F \subset C$
$\Rightarrow$ <u>Equivalent</u>
① F is closed in X.
② F is closed in C.

X MS

$F_\mu \subset X$ compact ($\mu \in M$)

$\bigcap_\mu F_\mu \neq \phi$

$\Rightarrow \bigcap_\mu F_\mu$: compact

<u>Proof.</u>

As F_μ is compact in a MS,
it is closed in X.

Consequently, $\bigcap_\mu F_\mu$ is also closed in X.

As $\bigcap_\mu F_\mu$ is closed in a compact set F_{μ^*},
it is compact.

//

cf.

$$\boxed{\begin{array}{l} (X, \mathfrak{G}) \text{ top. space} \\ F_\mu \subset X \text{ compact } (\mu \in M) \\ \bigcap_{\mu \in M} F_\mu \neq \emptyset \\ \not\Rightarrow \bigcap_{\mu} F_\mu : \text{compact} \end{array}}$$

ex.

$X = \mathbb{R}$

$\mathfrak{G} = \{\emptyset, [0, \lambda), [0, \infty), \mathbb{R} \mid \lambda > 0\}$

Then, $(X, \mathfrak{G})$ is a top. space.

Let $F_\mu = [-\mu, \infty)$ $(\mu > 0)$.

Then, $\bigcap_{\mu > 0} F_\mu = [0, \infty)$ $(\neq \emptyset)$.

- F_μ is compact ${}^\forall \mu > 0$.
- $\bigcap_{\mu > 0} F_\mu = [0, \infty)$ is not compact.

$(\because)$ Let $G_\lambda = [0, \lambda)$ $(\lambda > 0)$. Then,

- $\{G_\lambda\}_{\lambda > 0}$ is an open covering of $[0, \infty)$.
- ${}^\forall \{G_{\lambda_i}\}_{i=1}^N \subset \{G_\lambda\}$: $[0, \infty) \not\subset \bigcup_{i=1}^N G_{\lambda_i}$.

//

Compactness (2)

1. Aをコンパクト距離空間Xの無限個の要素を持つ部分集合とすると，Aは集積点を持つ．これを示せ．

2. $\{x_n\}$をコンパクト距離空間X内の点列とすると，ある$\{x_n\}$の部分列$\{x_{n_i}\}$と$x \in X$が存在し$x_{n_i} \to x$となる．なぜか？

3. Xをコンパクトではない距離空間とする．このとき，X内のある点列$\{x_n\}$については，どんな部分列$\{x_{n_i}\}$とXの要素xをとっても$x_{n_i} \to x$とできるとは限らない．このことを示す例をいくつか挙げよ．

4. コンパクト距離空間は完備である．このことを証明せよ．

5. Xを距離空間とする．問題4からXのコンパクトな部分集合AはXにおいて閉だとわかるが，Xのコンパクト部分集合Aは閉集合であることを（問題4を経由しないで）直接示せ．

6. 距離空間におけるコンパクト部分集合は有界であることを証明せよ．

7. 距離空間の部分集合について，コンパクト性，完備性，閉性，有界性の関係をベン図で描き，それぞれの領域に属す例を挙げよ．

8. $(X, \boldsymbol{G})$をコンパクトな位相空間，fをX上で定義された実数値連続関数とする．このとき，fはX上で最大値と最小値をとる．このことを証明せよ．

9. Fをコンパクト距離空間Xの空でない部分集合とする．このとき，FがコンパクトであることとXにおける閉集合であることは同値である．このことを示せ．また，Xがコンパクトではない場合やたとえコンパクトでも一般の位相空間の場合はこの同値性は成り立たないことを確認せよ．

10. 実数空間$\mathbb{R}$において，空でない部分集合Fがコンパクトであることは，Fが有界閉集合であることとして特徴付けられる．このことを示せ．

11. (おまけ) $\{F_\mu\}_{\mu \in M}$ $(M \neq \emptyset)$を距離空間Xの共通部分を持つコンパクト集合族とする．このとき，$\bigcap_{\mu \in M} F_\mu$もコンパクトであることを示せ．