Continuous mappings (1)

Def

X, Y MSs

$f : X \to Y$

- f : <u>continuous</u> at $x_0 \in X$.

  $\Leftrightarrow x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

- f : <u>continuous</u> (on X)

  $\Leftrightarrow {}^\forall x \in X$, f is continuous at x.

$$x_n \to x_0 \quad \Rightarrow \quad f(x_n) \to f(x_0)$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

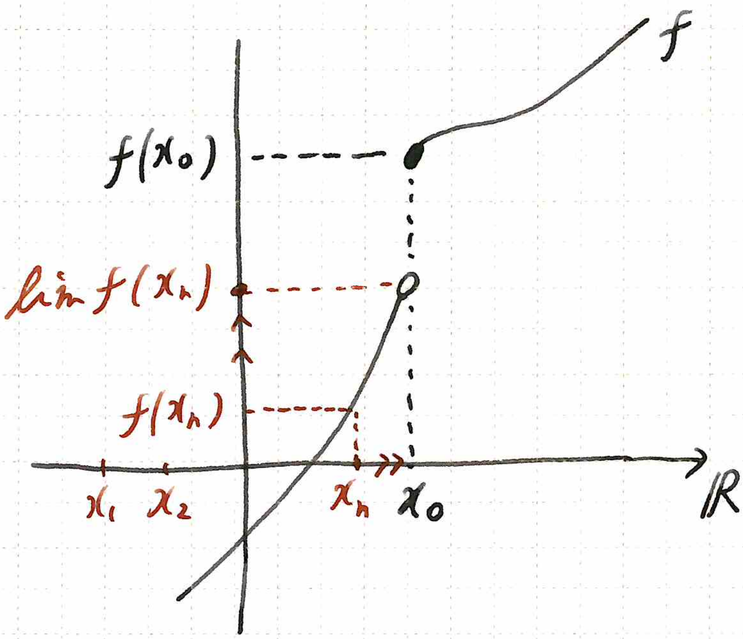
$$x_0 = \lim x_n \qquad\qquad f(x_0) = \lim f(x_n)$$

$$\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad f(\lim x_n)$$

$\therefore \lim f(x_n) = f(\lim x_n)$

map してから lim　　lim をとってからマップする



$$\exists \{x_n\} \subset X : \begin{cases} x_n \to x_0 \\ f(x_n) \not\to f(x_0) \end{cases}$$

$\therefore$ f is not continuous at x_0

X, Y, Z MSs
$f: X \to Y$ continuous
$g: Y \to Z$ "
$\Longrightarrow g \circ f: X \to Z$ continuous

Proof.

Let $x_0 \in X$ and $\{x_n\} \subset X : x_n \to x_0$.

We prove that $(g \circ f)(x_n) \to (g \circ f)(x_0)$.

As f is continuous and $x_n \to x_0$,

$$f(x_n) \to f(x_0) \text{ in } Y.$$

As g is continuous,

$$g(f(x_n)) \to g(f(x_0)).$$

This means that

$$(g \circ f)(x_n) \to (g \circ f)(x_0).$$

//

ex

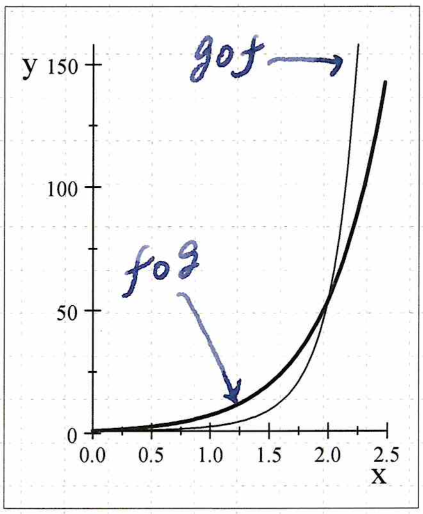
$X = Y = Z = \mathbb{R}$

$\left.\begin{array}{l} f(x) = x^2 \\ g(x) = e^x \end{array}\right\}$ continuous

Then, $(f \circ g)(x) = (e^x)^2 = e^{2x}$ and

$(g \circ f)(x) = e^{x^2}$.

Clearly, $f \circ g$ and $g \circ f$ are continuous.



Cor

$\boxed{\begin{array}{l} X, Y, Z, W \quad \text{MSs} \\ \left.\begin{array}{l} f: X \to Y \\ g: Y \to Z \\ h: Z \to W \end{array}\right\} \text{continuous} \\ \Longrightarrow h \circ g \circ f : X \to W \text{ continuous} \end{array}}$

X, Y MSs

$f: X \to Y$ constant

i.e. $\exists y_0 \in Y : \forall x \in X, f(x) = y_0$

$\Rightarrow f$: continuous

Proof.

Let $x \in X$, $x_n \to x$.

We prove that $f(x_n) \to f(x)$.

As $f(x_n) = y_0$ ($\forall n \in \mathbb{N}$), we have

$$f(x_n) \to y_0 = f(x).$$

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Our next purpose :

Th

X, Y MSs

$f : X \to Y$

$x_0 \in X$

$\Rightarrow$ <u>Equivalent</u>

① f : continuous at x_0

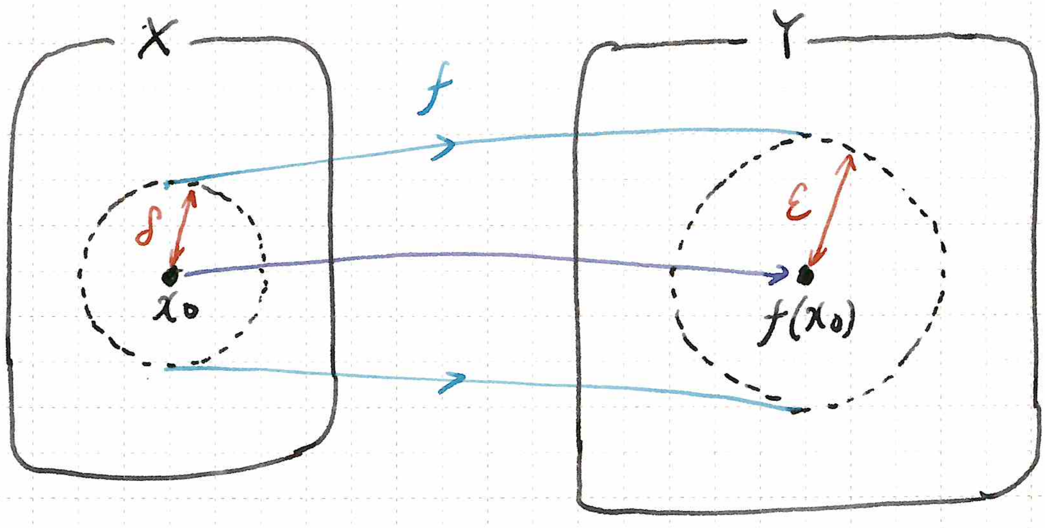
i.e. $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

② $\forall \varepsilon > 0, \exists \delta > 0$:

$d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$

※ $\underline{d(x, x_0) < \delta} \Rightarrow \underline{d(f(x), f(x_0)) < \varepsilon}$

$\Leftrightarrow x \in S_\delta(x_0)$ $\qquad$ $\Leftrightarrow f(x) \in S_\varepsilon(f(x_0))$



Th

X, Y MSs
$f: X \to Y$
$x_0 \in X$
$\Rightarrow$ Equivalent

① f: continuous at x_0
i.e. $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

② $\forall \varepsilon > 0, \exists \delta > 0:$
$$d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon.$$

Proof

① $\Rightarrow$ ②

Suppose by way of contradiction that
$$\exists \varepsilon > 0 : \forall \delta > 0, \exists x_\delta \in X : \begin{cases} d(x_\delta, x_0) < \delta \\ d(f(x_\delta), f(x_0)) \geqq \varepsilon . \end{cases}$$

Letting $\delta = 1/n > 0$, we obtain
$$\exists \varepsilon > 0, \{x_n\} \subset X : \begin{cases} d(x_n, x_0) < \frac{1}{n} \\ d(f(x_n), f(x_0)) \geqq \varepsilon . \end{cases}$$

$$\therefore \exists \{x_n\} \subset X : \begin{cases} x_n \to x_0 \\ f(x_n) \not\to f(x_0) . \end{cases}$$

This contradicts ①. $\lrcorner$

② $\Rightarrow$ ①

Let $(x_n) \subset X : x_n \to x_0$.

We prove that $f(x_n) \to f(x_0)$.

i.e. $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}:$

$$n \geqq n_0 \Rightarrow d(f(x_n), f(x_0)) < \varepsilon$$

Let $\varepsilon > 0$.

From ②, for $\varepsilon > 0$,

$\exists \delta > 0 :$

$$d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon. \quad -(*)$$

As $x_n \to x_0$, for $\delta > 0$,

$$\exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow d(x_n, x_0) < \delta. \quad -(**)$$

From $(*)$ and $(**)$,

$$n \geqq n_0 \xRightarrow[(**)]{} d(x_n, x_0) < \delta$$

$$\xRightarrow[(*)]{} d(f(x_n), f(x_0)) < \varepsilon.$$

$\therefore \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}:$

$$n \geqq n_0 \Rightarrow d(f(x_n), f(x_0)) < \varepsilon.$$

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②$\Rightarrow$① (alternative proof)

Let $\{x_n\} \subset X : x_n \to x_0$.

Suppose by way of contradiction that

$$f(x_n) \not\to f(x_0).$$

Then, $\exists \varepsilon > 0,\ \{x_{n_i}\} \subset \{x_n\}$:

$$\forall i \in \mathbb{N},\ d(f(x_{n_i}), f(x_0)) \geqq \varepsilon. \quad -(*)$$

From ②, for $\varepsilon > 0$,

$$\exists \delta > 0 : d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon. \quad -(**)$$

As $x_n \to x_0$, we have $x_{n_i} \to x_0$.

For $\delta > 0$, $\exists i_0 \in \mathbb{N} : i \geqq i_0 \Rightarrow d(x_{n_i}, x_0) < \delta$.

From $(**)$, $d(f(x_{n_i}), f(x_0)) < \varepsilon \quad (i \geqq i_0)$.

This contradicts $(*)$.

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ex

$f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = ax \quad {}^\forall x \in \mathbb{R}$$

$\Rightarrow f$: continuous

$(\because)$

Assume, w.l.g., that $a \neq 0$.

Let $x_0 \in \mathbb{R}$.

We show that

$${}^\forall \varepsilon > 0, {}^\exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Let $\varepsilon > 0$.

Define $\delta = \varepsilon / |a| > 0$.

Choose $x \in \mathbb{R} : |x - x_0| < \delta$.

Then, we have

$$\begin{aligned} &|f(x) - f(x_0)| \\ &\quad = |ax - ax_0| \\ &\quad = |a||x - x_0| \\ &\quad < |a|\delta = \varepsilon. \end{aligned}$$

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Review

Def

X MS

$f, g : X \to \mathbb{R}$

$\alpha \in \mathbb{R}$

- $|f|(x) := |f(x)| \quad \forall x \in X$
- $(f+g)(x) := f(x) + g(x) \quad \forall x \in X$
- $(fg)(x) := f(x) \cdot g(x) \quad \forall x \in X$
- $(\alpha f)(x) := \alpha \cdot f(x) \quad \forall x \in X$
- $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)} \quad \forall x \in X$

Th

$\{a_n\}, \{b_n\} \subset \mathbb{R} : a_n \to a, \ b_n \to b$

$\alpha \in \mathbb{R}$

$\Rightarrow$ (1) $|a_n| \to |a|$

(2) $a_n + b_n \to a + b$

(3) $a_n b_n \to ab$

(4) $\alpha a_n \to \alpha a$

(5) $\frac{a_n}{b_n} \to \frac{a}{b}$

where $b_n, b \neq 0$

X MS
$f, g : X \to \mathbb{R}$ continuous
$\alpha \in \mathbb{R}$
$\Rightarrow$ (1) $|f|$: continuous
(2) $f+g$: 〃
(3) $f \cdot g$: 〃
(4) αf : 〃
(5) $\frac{f}{g}$: 〃
where $g(x) \neq 0 \ \forall x \in X$

Proof

(1) Let $x_n \to x$, where $x \in X$.
We prove that $|f|(x_n) \to |f|(x)$.
i.e. $|f(x_n)| \to |f(x)|$
As $x_n \to x$ and f is continuous,
$f(x_n) \to f(x)$.
Therefore, we have $|f(x_n)| \to |f(x)|$. $\lrcorner$

(2) Let $x_n \to x \in X$.

Our goal is to show that

$$\underline{(f+g)(x_n) \to (f+g)(x).}$$

i.e. $f(x_n) + g(x_n) \to f(x) + g(x)$

As $x_n \to x$ and f, g are continuous,

we have $\begin{cases} f(x_n) \to f(x) \\ g(x_n) \to g(x) \end{cases}$.

Therefore,

$$f(x_n) + g(x_n) \to f(x) + g(x).$$

In much the same way, we can easily verify (3) - (5).

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Remark

$f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = ax^2 + bx + c \quad \forall x \in \mathbb{R}$$

$\Rightarrow f$: continuous

Appendix

- $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

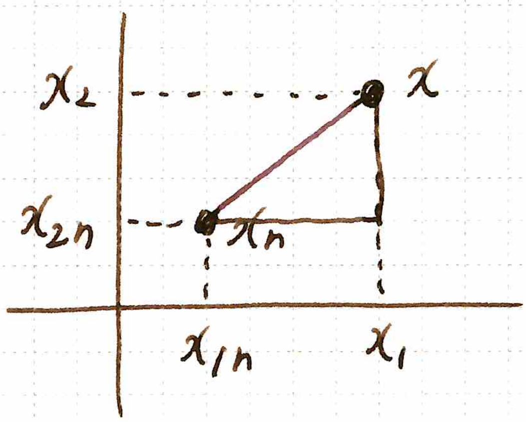
$$\Leftrightarrow \forall x \in \mathbb{R}^2,\ \underset{\mathbb{R}^2}{\underset{\cap}{x_n}} \to \underset{\mathbb{R}^2}{\underset{\cap}{x}} \Rightarrow \underset{\mathbb{R}}{\underset{\cap}{f(x_n)}} \to \underset{\mathbb{R}}{\underset{\cap}{f(x)}}.$$

$$x_n \equiv (x_{1n}, x_{2n})$$

$$x \equiv (x_1, x_2)$$

Then, $d(x_n, x)$

$$= \sqrt{(x_{1n} - x_1)^2 + (x_{2n} - x_2)^2}$$



- $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous

$$\Leftrightarrow {}^{\forall} x \in \mathbb{R}^3,\ \underset{\underset{\mathbb{R}^3}{\cap}}{x_n} \to \underset{\underset{\mathbb{R}^3}{\cap}}{x} \Rightarrow \underset{\underset{\mathbb{R}}{\cap}}{f(x_n)} \to \underset{\underset{\mathbb{R}}{\cap}}{f(x)}$$

$$x_n = (x_{1n}, x_{2n}, x_{3n})$$

$$x = (x_1, x_2, x_3)$$

Then, $d(x_n, x)$

$$= \sqrt{(x_{1n} - x_1)^2 + (x_{2n} - x_2)^2 + (x_{3n} - x_3)^2}$$

- $f : \mathbb{R}^M \to \mathbb{R}^N$

It is tedious !

Continuous mappings (1)

1. X, Yを距離空間, $f : X \to Y$, $x_0 \in X$とする.
(1) fがx_0において連続であること,
(2) fがXにおいて連続であること
の定義を述べ, グラフを用いて説明せよ.

2. X, Yを距離空間, $f : X \to Y$, $x_0 \in X$とする.
(1) fがx_0において連続ではないこと,
(2) fがXにおいて連続ではないこと
について, 論理式とグラフを用いて説明せよ.

3. X, Y, Zを距離空間, 2つの写像$f : X \to Y$, $g : Y \to Z$を連続とすると, これらの合成写像$g \circ f : X \to Z$も連続である. このことを, わざわざ証明を書くまでもなく納得できるまで考えてみよ.

4. 定値写像は連続である. このことを, わざわざ証明を書くまでもなく納得できるまで考えてみよ.

5. X, Yを距離空間, また$f : X \to Y$, $x_0 \in X$とする. このとき, fがx_0で連続であることと条件

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$$

が同値であることを証明せよ.

6. 実数値関数$f(x) = a|x|$が連続であることを$\varepsilon - \delta$論法を用いて証明せよ. ここで, aは実数である.

7. Xを距離空間, $f, g : X \to \mathbb{R}$を連続関数とする, また$\alpha \in \mathbb{R}$とする. このとき,

$$(1)\ |f|, \quad (2)\ f + g, \quad (3)\ fg, \quad (4)\ \alpha f, \quad (5)\ \frac{f}{g}$$

も連続関数である. このことを示せ. ただし, gは(5)については, $0 (\in \mathbb{R})$に値を取らないことは前提とする.

8. 問題7の設定で, $|f|$は連続だがfは連続ではない例を挙げよ. 同様に, $f + g$についても考えてみよ.

9. 実数値連続関数$f : \mathbb{R} \to \mathbb{R}$を考える. このとき,

$$F(x) = af^2(x) + bf(x) + c$$

として定義される関数Fを考える. ただし, $f^2(x)$はfを2回合成した関数, すなわち$f^2(x) = f(f(x))$である. このとき, Fは連続である. なぜか?