Cauchy sequences and complete metric spaces

Def.

X metric space

$\{x_n\} \subset X$ Cauchy seq.

$\Leftrightarrow m, n \to \infty \Rightarrow d(x_m, x_n) \to 0$

$\Leftrightarrow {}^\forall \varepsilon > 0, {}^\exists n_0 \in \mathbb{N}:$

$m, n \geqq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$

ex

$\{x_n\} = \left\{\frac{1}{n}\right\} \subset \mathbb{R}$

Then, • $\{x_n\}$ is a Cauchy seq. in $\mathbb{R}$.

• $\{x_n\}$ is not a Cauchy seq.

in $\mathbb{R}$ with the discrete metric

ex

$\{3, 3.1, 3.14, 3.141, \cdots\} \subset \mathbb{R}$

Cauchy sequence

X metric space

$\{x_n\} \subset X$ convergent

i.e. $\exists x \in X : x_n \to x$

$\Rightarrow \{x_n\}$: Cauchy seq.

<u>Proof</u>

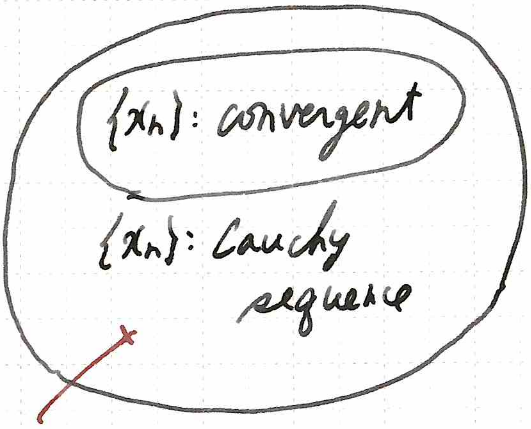
Let $m, n \in \mathbb{N}$.

It holds that

$$
\begin{aligned}
d(x_m, x_n) & \\
&\leq d(x_m, x) + d(x, x_n) \\
&\to 0.
\end{aligned}
$$

Thus, $d(x_m, x_n) \to 0$ as $m, n \to \infty$.

//



$X = (0, 1]$

$\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

Def.

$$\boxed{\begin{array}{l} X \text{ MS} \\ X \ \underline{\text{complete}} \\ \Longleftrightarrow \{x_n\} \subset X \text{ Cauchy seq.} \\ \qquad \Rightarrow \{x_n\} \subset X \text{ convergent} \end{array}}$$

ex

- $\mathbb{R}$: complete
  (We show this fact in what follows.)
- $(0, 1] \ (\subset \mathbb{R})$: not complete
- $\mathbb{Q} \ (\subset \mathbb{R})$: not complete

(X, d) MS

with the discrete metric

i.e. $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

$\Rightarrow (X, d)$ complete

X MS
$\{x_n\} \subset X$ Cauchy sequence
$\Rightarrow \{x_n\}$: bdd

Proof.
We prove that
$\exists x \in X, M > 0 : \forall n \in \mathbb{N}, d(x_n, x) \leqq M$.

As $\{x_n\} \subset X$ is a Cauchy sequence,
$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : m, n \geqq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$.

Therefore, $d(x_m, x_{n_0}) < \varepsilon \quad (\forall m \geqq n_0)$. — (*)
Let $M = \max\{d(x_1, x_{n_0}), \cdots, d(x_{n_0-1}, x_{n_0}), \varepsilon\} > 0$.
Let $n \in \mathbb{N}$.
Then, from (*), $d(x_n, x_{n_0}) \leqq M$

$\therefore \exists x_{n_0} \in X, M > 0$:
$\forall n \in \mathbb{N}, d(x_n, x_{n_0}) \leqq M$.

This means that $\{x_n\}$ is bdd.
//

X MS
$\{x_n\} \subset X$ Cauchy sequence
$\exists \{x_{n_i}\} \subset \{x_n\},\ x \in X : x_{n_i} \to x$
$\Longrightarrow x_n \to x$

Proof.

We show that

$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow d(x_n, x) < \varepsilon.$

Let $\varepsilon > 0$.

As $\{x_n\}$ is a Cauchy sequence, for $\varepsilon/2 > 0$,

$\exists n_0 \in \mathbb{N} : m, n \geqq n_0 \Rightarrow d(x_m, x_n) < \frac{\varepsilon}{2}.$ — ①

As $x_{n_i} \to x$, for $\varepsilon/2 > 0$,

$\exists i_0 \in \mathbb{N} : i \geqq i_0 \Rightarrow \begin{cases} n_i \geqq n_0 & \text{— ②} \\ d(x_{n_i}, x) < \frac{\varepsilon}{2}. & \text{— ③} \end{cases}$

From ②, $n_{i_0} \geqq n_0$. — ④

From ③, $d(x_{n_{i_0}}, x) < \frac{\varepsilon}{2}$. — ⑤

Let $n \geqq n_0$.

From ④ and ①, $d(x_{n_{i_0}}, x_n) < \frac{\varepsilon}{2}$. — ⑥

It holds that

$$d(x_n, x) \leqq d(x_n, x_{n_{i_0}}) + d(x_{n_{i_0}}, x) \quad ⑤⑥$$
$$< \varepsilon.$$

$\therefore \forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow d(x_n, x) < \varepsilon.$

//

Th

$\boxed{\mathbb{R} : \text{complete}}$

Proof

Let $\{x_n\} \subset \mathbb{R}$ be a Cauchy seq.

Then, $\{x_n\}$ is bdd, and consequently,

$$\exists \{x_{n_i}\} \subset \{x_n\},\ x \in \mathbb{R} : x_{n_i} \to x.$$

We obtain $x_n \to x$.

//

Remark

Axiom

$A \subset \mathbb{R}$ bdd above

$\Longrightarrow \exists \alpha = \sup A \in \mathbb{R}$

Th

$\{a_n\} \subset \mathbb{R}$ bdd above
monotone increasing

$\Longrightarrow \{a_n\}$: convergent

i.e. $\exists a \in \mathbb{R} : a_n \to a$

Th (Bolzano-Weierstrass)

$\{a_n\} \subset \mathbb{R}$ bdd

$\Longrightarrow \exists \{a_{n_i}\} \subset \{a_n\},\ a \in \mathbb{R} : a_{n_i} \to a$

Th

$\mathbb{R}$: complete

これらは全て$\mathbb{R}$の完備性の表現.

Th

> $\{a_n\} \subset \mathbb{R}$ bdd above,
> monotone increasing
> $\Longrightarrow \{a_n\}$: convergent

Proof

It suffices to prove that
<u>$\{a_n\}$ is a Cauchy sequence.</u>

Suppose to lead a contradiction that

$\exists \varepsilon > 0 : \forall n \in \mathbb{N},$

$\exists n' > n : |a_{n'} - a_n| \geqq \varepsilon. \quad — (*)$

Let $n_0 \in \mathbb{N}$.

From (*), for $n = n_0 \in \mathbb{N}$,

$\exists n_1 > n_0 : |a_{n_1} - a_{n_0}| \geqq \varepsilon.$

As $\{a_n\}$ is monotone increasing, $a_{n_0} \leqq a_{n_1}$.

Therefore, $a_{n_0} + \varepsilon \leqq a_{n_1}. \quad — (**)$

From (*), for $n = n_1 \in \mathbb{N}$,

$\exists n_2 > n_1 : \ |a_{n_2} - a_{n_1}| \geqq \varepsilon$.

Hence, $a_{n_1} + \varepsilon \leqq a_{n_2}$.

From (**), $a_{n_0} + 2\varepsilon \leqq a_{n_1} + \varepsilon$

$\leqq a_{n_2}$.

Using this method, we obtain

$\{a_{n_k}\} \subset \{a_n\} : a_{n_0} + k\varepsilon \leqq a_{n_k}$

As $k \to \infty$, $a_{n_k} \to \infty$.

This contradicts the assumption that $\{a_n\}$ is bdd above.

//

X metric space

- $\{x_n\} \subset X$ convergent

$\Longleftrightarrow \exists x \in X : \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow d(x_n, x) < \varepsilon$

全空間の情報

- $\{x_n\} \subset X$ Cauchy seq.

$\Longleftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : m, n \geqq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$

※全空間の情報は入っていない。

X metric space
$A \subset X$
$\{x_n\} \subset A \ (\subset X)$
$\Rightarrow$ Equivalent
① $\{x_n\} \subset X$ Cauchy
② $\{x_n\} \subset A$ Cauchy

収束の場合は,

X metric space

$A \subset X$

$\{x_n\} \subset A \ (\subset X)$

$\{x_n\}$: convergent in A.

$\overset{\Longrightarrow}{\not\Longleftarrow}$ $\{x_n\}$: convergent in X.

ex

$X = \mathbb{R}$

$A = (0, 1]$

$\{x_n\} = \{\frac{1}{n}\} \subset A \subset \mathbb{R}$.

Then, · $\{x_n\}$ is convergent in $\mathbb{R}$.

· $\{x_n\}$ is not convergent in A.

X metric space
$A \subset X$ complete
$\Rightarrow A$: closed in X.

Proof

Let $\{x_n\} \subset A : x_n \to x \in X$.

We prove that $x \in A$.

As $\{x_n\} (\subset A) \subset X$ is convergent in X,
it is a Cauchy seq. in X.

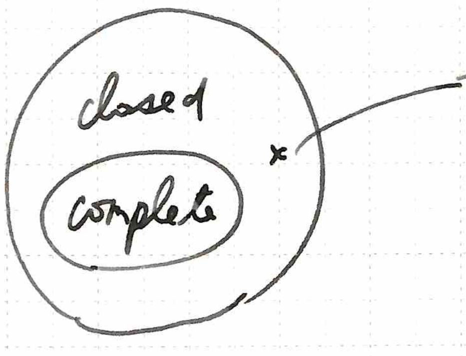
Thus, $\{x_n\}$ is a Cauchy seq. in A.

As A is complete, $\exists y \in A : x_n \to y$.

As $x_n \to x$ and $x_n \to y$, we have $x = y$.

Therefore, $x (= y) \in A$.

//



ex

$X = (0, \infty)$

$A = (0, 1]$

Then, A is closed in X.

However, A is not complete.

Th

X complete metric space

$A \subset X$

$\Rightarrow$ Equivalent

① A : complete

② $A (\subset X)$: closed.

Proof

①$\Rightarrow$② OK.

②$\Rightarrow$①

Let $\{x_n\} \subset A$ be a Cauchy seq.

We show that $\exists x \in A : x_n \to x$.

As $\{x_n\} (\subset A) \subset X$, it is a Cauchy seq., and

X is complete,

$\exists x \in X : x_n \to x$.

As $\{x_n\} \subset A$ and $x_n \to x \in X$,

it follows from ② that $x \in A$.

Therefore, $\exists x \in A : x_n \to x$.

//

ex

- $\mathbb{R}$ complete
- $[0,1]$ complete
- $\mathbb{N}$ complete
- $(0,1]$ not complete
- $\mathbb{Q}$ not complete
- $[0,1] \cup [2,3]$ complete
- $[0,2] \setminus \{1\}$ not complete

Cauchy sequences and complete metric spaces

1. 距離空間におけるCauchy列の定義を述べ，例を挙げて説明せよ．

2. 距離空間において，収束する数列はCauchy列である．このことを示せ．また，逆が言えないことを示す例を挙げよ．

3. 離散距離空間は完備である．なぜか？

4. 距離空間においてCauchy列は有界であることを示せ．

5. Xを距離空間，$\{x_n\}$をXのCauchy列とする．点列$\{x_n\}$が収束する部分列を持つならば，$\{x_n\}$自体もその極限に収束することをを示せ．

6. 実数空間$\mathbb{R}$の完備性を証明せよ．

7. Xを距離空間，AをXの完備な部分集合とする．このとき，Aは閉集合である（完備ならば閉）．このことを示せ．

8. 距離空間の部分集合で閉集合だが完備ではないものの例を挙げよ．

9. Xを完備距離空間，AをXの部分集合とする．このとき，Aが完備であることと（Xにおいて）閉であることは同値である．このことを証明せよ．

10. 問題6と9より，$\mathbb{R}$の閉部分集合は完備であり，閉でない部分集合は完備ではない．このことを用いて，完備距離空間と完備ではない距離空間の例を2つずつ挙げよ（$\mathbb{R}$の部分集合でよい）．