Limit of a mapping

Def.

X, Y MSs

$x_0 \in X$

$f : X \setminus \{x_0\} \to Y$

- $x \to x_0 \Rightarrow f(x) \to y_0$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 :$

$0 < d(x, x_0) < \delta \Rightarrow d(f(x), y_0) < \varepsilon$

関数の極限

$\Leftrightarrow f(x) \to y_0$ (as $x \to x_0$)

ex

$X = [0, \infty)$

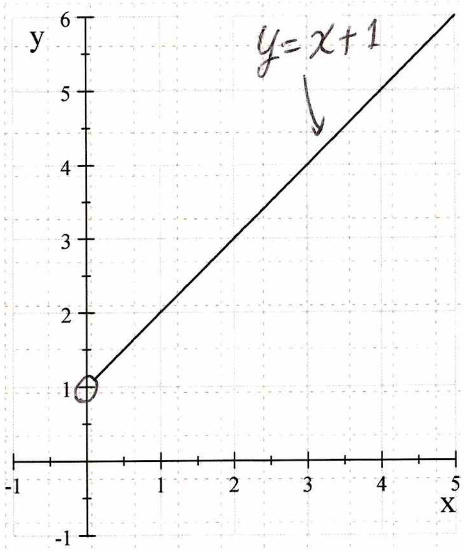
$Y = \mathbb{R}$

$x_0 = \{0\}$

$f : X \setminus \{x_0\} \to Y$ defined by

$$f(x) = x + 1 \quad \forall x \in (0, \infty)$$

Then, $f(x) \to 1$ as $x \to 0$.



ex

$X \subset \mathbb{R}$

$Y = \mathbb{R}$

$f : X \to \mathbb{R}$

Define $F : X \setminus \{0\} \to \mathbb{R}$ as follows:

$$F(h; x) = \frac{f(x+h) - f(x)}{h}$$

In this case,

$$\boxed{\begin{aligned} &h \to 0 \Rightarrow F(h; x) \to \alpha \\ &\Leftrightarrow \alpha = f'(x) \end{aligned}}$$

X, Y MSs
$x_0 \in X$
$f : X \setminus \{x_0\} \to Y$
$f(x) \to y$ as $x \to x_0$ — ①
$f(x) \to z$ as $x \to x_0$ — ②
$\Rightarrow y = z$

Proof

Suppose to lead a contradiction that

$y \neq z.$

Let $\varepsilon = \frac{1}{2} d(y, z) > 0.$

From ①, for $\varepsilon > 0$,

$\exists \delta_1 > 0 : 0 < d(x, x_0) < \delta_1 \Rightarrow d(f(x), y) < \varepsilon.$

From ②, for $\varepsilon > 0$,

$\exists \delta_2 > 0 : 0 < d(x, x_0) < \delta_2 \Rightarrow d(f(x), z) < \varepsilon.$

Define $\delta = \min\{\delta_1, \delta_2\} > 0.$

If $0 < d(x, x_0) < \delta \; (\leqq \delta_1, \delta_2)$, then

$2\varepsilon = d(y, z) \leqq d(y, f(x)) + d(f(x), z) < 2\varepsilon.$

This is a contradiction. //

※ $x \to x_0 \Rightarrow f(x) \to y$

$\Leftrightarrow y = \lim_{x \to x_0} f(x)$

X MS

$x_0 \in X$

$f, g, h : X \setminus \{x_0\} \to \mathbb{R}$

$f(x) \leq g(x) \leq h(x) \quad \forall x \in X \setminus \{x_0\}$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = a$$

$$\Longrightarrow \lim_{x \to x_0} g(x) = a$$

X MS
$x_0 \in X$
$f, g : X \setminus \{x_0\} \to \mathbb{R}$
$f(x) \leqq g(x) \quad {}^\forall x \in X \setminus \{x_0\}$
$f(x) \to a \quad (x \to x_0) \quad$ — ①
$g(x) \to b \quad (x \to x_0) \quad$ — ②
$\Longrightarrow a \leqq b$

<u>Proof</u>

Suppose by contradiction that $a > b$.

Define $\varepsilon = \frac{1}{3}(a-b) > 0$.

From ①, for $\varepsilon > 0$,

$\exists \delta_1 > 0 : 0 < d(x, x_0) < \delta_1 \Rightarrow |f(x) - a| < \varepsilon$.

From ②, for $\varepsilon > 0$,

$\exists \delta_2 > 0 : 0 < d(x, x_0) < \delta_2 \Rightarrow |g(x) - b| < \varepsilon$.

Define $\delta = \min\{\delta_1, \delta_2\} > 0$.

If $0 < d(x, x_0) < \delta \ (\leqq \delta_1, \delta_2)$, then

$$\begin{aligned} 3\varepsilon &= a - b \\ &= a - f(x) + \underbrace{f(x) - g(x)}_{\leqq 0} + g(x) - b \\ &\leqq |a - f(x)| + |g(x) - b| < 2\varepsilon. \end{aligned}$$

This is a contradiction. //

Lemma

X MS

$x_0 \in X$

$f : X \setminus \{x_0\} \to \mathbb{R}$

$f(x) \to a \ (x \to x_0)$

i.e. $\forall \varepsilon > 0, \exists \delta > 0 :$

$0 < d(x, x_0) < \delta \Rightarrow |f(x) - a| < \varepsilon$ —(*)

$\Rightarrow \exists \delta > 0, M > 0 :$

$0 < d(x, x_0) < \delta \Rightarrow |f(x)| \leq M$

Proof

From (*), for $\varepsilon = 1 > 0$, $\exists \delta > 0 :$

$0 < d(x, x_0) < \delta \Rightarrow |f(x) - a| < 1.$

Therefore,

$$|f(x)| \leq |f(x) - a| + |a| < 1 + |a| \equiv M.$$

We obtain the desired result. //

cf.

$$\boxed{\begin{array}{l} f(x) \to a \ (x \to x_0) \\ \not\Rightarrow \exists M > 0 : \forall x \in X \setminus \{x_0\}, \\ \quad |f(x)| \leq M \end{array}}$$

ex

$f(x) = x$

$f(x) \to 0 \quad (x \to 0)$

However,

$\forall M > 0, \exists x \in \mathbb{R} \setminus \{0\} : |f(x)| > M.$

Th

X MS

$x_0 \in X$

$f, g : X \setminus \{x_0\} \to \mathbb{R}$

$f(x) \to \alpha$, $g(x) \to \beta$ as $x \to x_0$

$\Rightarrow$ ① $|f(x)| \to |\alpha|$

② $f(x) + g(x) \to \alpha + \beta$

③ $f(x)\, g(x) \to \alpha \beta$

④ $\dfrac{f(x)}{g(x)} \to \dfrac{\alpha}{\beta}$

as $x \to x_0$,

where $g(x), \beta \neq 0$ in ④.

Proof

②: We show that

$\forall \varepsilon > 0, \exists \delta > 0:$

$0 < d(x, x_0) < \delta \Rightarrow \left| (f(x) + g(x)) - (\alpha + \beta) \right| < \varepsilon.$

Let $\varepsilon > 0$.

As $f(x) \to \alpha$ (as $x \to x_0$),

for $\varepsilon/2 > 0$, $\exists \delta_1 > 0$:

$0 < d(x, x_0) < \delta_1 \Rightarrow |f(x) - \alpha| < \frac{\varepsilon}{2}$. —(*)

As $g(x) \to \beta$ (as $x \to x_0$),

for $\varepsilon/2 > 0$, $\exists \delta_2 > 0$:

$0 < d(x, x_0) < \delta_2 \Rightarrow |g(x) - \beta| < \frac{\varepsilon}{2}$. —(**)

Let $\delta = \min\{\delta_1, \delta_2\} > 0$.

Let $x \in X$: $0 < d(x, x_0) < \delta$ $(\leq \delta_1, \delta_2)$.

Then, $\left| (f(x) + g(x)) - (\alpha + \beta) \right|$

$$\leq |f(x) - \alpha| + |g(x) - \beta|$$

(*) (**)

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

∎

③: We show that

$\forall \varepsilon > 0, \exists \delta > 0:$

$0 < d(x, x_0) < \delta \Rightarrow |(fg)(x) - \alpha\beta| < \varepsilon.$

As $g(x) \to \beta$ $(x \to x_0)$,

$\exists \delta_0 > 0;\ M > 0:$

$0 < d(x, x_0) < \delta_0 \Rightarrow |g(x)| \leq M.$

If $0 < d(x, x_0) < \delta_0$, then

$$
\begin{aligned}
&|(fg)(x) - \alpha\beta| \\
&\leq |f(x)g(x) - \alpha g(x)| + |\alpha g(x) - \alpha\beta| \\
&= |g(x)|\,|f(x) - \alpha| + |\alpha|\,|g(x) - \beta|. \quad -(*)
\end{aligned}
$$

Let $\varepsilon > 0$.

As $f(x) \to \alpha$ $(x \to x_0)$, $\exists \delta_1 > 0:$

$0 < d(x, x_0) < \delta_1 \Rightarrow |f(x) - \alpha| < \dfrac{\varepsilon}{2M}. \quad -(**)$

As $g(x) \to \beta$ $(x \to x_0)$, $\exists \delta_2 > 0:$

$0 < d(x, x_0) < \delta_2 \Rightarrow |g(x) - \beta| < \dfrac{\varepsilon}{2|\alpha|}. \quad -(*3)$

(i) $\alpha = 0$

Let $\delta = \min\{\delta_0, \delta_1\} > 0$.

If $0 < d(x, x_0) < \delta \ (\leqq \delta_0, \delta_1)$, then

$$|(fg)(x) - \alpha\beta|$$

$$\leqq M\,|f(x) - \alpha| < M \cdot \frac{\varepsilon}{2M} = \varepsilon.$$

(ii) $\alpha \neq 0$

Let $\delta = \min\{\delta_0, \delta_1, \delta_2\} > 0$.

If $0 < d(x, x_0) < \delta \ (\leqq \delta_0, \delta_1, \delta_2)$,

then

$$|(fg)(x) - \alpha\beta|$$

$$\leqq M|f(x) - \alpha| + |\alpha|\,|g(x) - \beta|$$

$$< M \cdot \frac{\varepsilon}{2M} + |\alpha| \cdot \frac{\varepsilon}{2|\alpha|}$$

$$= \varepsilon.$$

$$\boxed{\begin{array}{l} g(x) \to \beta \ (x \to x_0) \\ g(x),\ \beta \neq 0 \\ \Rightarrow \dfrac{1}{g(x)} \to \dfrac{1}{\beta} \ (x \to x_0) \end{array}}$$

<u>Proof</u>

Assume for simplicity that $\beta > 0$.

As $g(x) \to \beta > 0$ $(x \to x_0)$,

$\exists \delta_1 > 0 : 0 < d(x, x_0) < \delta_1 \Rightarrow g(x) > \dfrac{\beta}{2}$.

Let $\varepsilon > 0$.

As $g(x) \to \beta$, for $\dfrac{\beta^2}{2}\varepsilon > 0$,

$\exists \delta_2 > 0 : 0 < d(x, x_0) < \delta_2 \Rightarrow |g(x) - \beta| < \dfrac{\beta^2}{2}\varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\} > 0$.

If $0 < d(x, x_0) < \delta$, then

$$\begin{aligned} \left| \frac{1}{g(x)} - \frac{1}{\beta} \right| &= \left| \frac{\beta - g(x)}{g(x)\beta} \right| \\ &= \frac{1}{g(x)\beta} |g(x) - \beta| \\ &< \frac{2}{\beta^2} \cdot \frac{\beta^2}{2} \varepsilon = \varepsilon. \end{aligned}$$

//

④ : $$\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$$

③

$$\longrightarrow \alpha \cdot \frac{1}{\beta} .$$

//

Th

X, Y MSs

$x_0 \in X$

$f : X \to Y$

$\Rightarrow$ Equivalent

① f : continuous at $x_0 \in X$

i.e. $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

② $\forall \varepsilon > 0, \exists \delta > 0$:

$d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$

③ $x \to x_0 \Rightarrow f(x) \to f(x_0)$

i.e. $\forall \varepsilon > 0, \exists \delta > 0$:

$0 < d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$

Proof

① $\Leftrightarrow$ ② OK

② $\Rightarrow$ ③ Obvious.

③ $\Rightarrow$ ②

For $\varepsilon > 0$, let $\delta > 0$ be that in ③.

Assume that $d(x, x_0) < \delta$.

If $0 < d(x, x_0)$, then from ③,

$d(f(x), f(x_0)) < \varepsilon$.

If $0 = d(x, x_0)$, then

$d(f(x), f(x_0)) = 0 < \varepsilon$.

//

※ ③ $\Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$

$= f(\lim x)$

※ In this theorem, it is supposed that

$f: X \to Y$ and $x_0 \in X$

$\therefore f(x_0)\ (\in Y)$ exists.

Appendix

Def.

$X = (a, \infty)$, where $a \in \mathbb{R}$

Y MS

$f : X \to Y$

- $x \to \infty \Rightarrow f(x) \to y_0$

$\Leftrightarrow \forall \varepsilon > 0, \exists M > 0 :$

$$x \geqq M \Rightarrow d(f(x), y_0) < \varepsilon$$

$\Leftrightarrow f(x) \to y_0 \ (\text{as } x \to \infty)$

ex.

- $f(x) = \frac{1}{x}$

  Then, $f(x) \to 0 \ (\text{as } x \to \infty)$

- $f(x) = e^{-x}$

  Then, $f(x) \to 0 \ (\text{as } x \to \infty)$.

$$\boxed{\begin{array}{l} X=(a,\infty),\text{ where } a\in\mathbb{R} \\ f,g: X\to\mathbb{R} \\ f(x)\leqq g(x)\quad \forall x\in X \\ f(x)\to\alpha\quad\text{as } x\to\infty \\ g(x)\to\beta\quad\text{as } x\to\infty \\ \Rightarrow \alpha\leqq\beta \end{array}}$$

<u>Proof</u>

Suppose to lead a contradiction that

$$\alpha > \beta.$$

Let $\varepsilon = \frac{1}{3}(\alpha-\beta) > 0$.

As $f(x)\to\alpha\ (x\to\infty)$, $\exists M_1 > 0$:

$$x \geqq M_1 \Rightarrow \alpha-\varepsilon < f(x) < \alpha+\varepsilon.$$

As $g(x)\to\beta\ (x\to\infty)$, $\exists M_2 > 0$:

$$x \geqq M_2 \Rightarrow \beta-\varepsilon < g(x) < \beta+\varepsilon.$$

Let $M = \max\{M_1, M_2\} > 0$

and choose $x \geqq M\ (\geqq M_1, M_2)$.

Then, $f(x) \leqq g(x) < \beta+\varepsilon$

$$< \alpha-\varepsilon < f(x).$$

This is a contradiction.

//

Limit of mapping

1. X, Yを距離空間, x_0をXの要素とする. また, $f : X \backslash \{x_0\} \to Y$とする. 写像の極限 $f(x) \to y_0$ (as $x \to x_0$) の定義を述べ, 例を挙げて説明せよ. また, なぜfの定義域として$\{x_0\}$を外しておくのか考えよ.

2. X, Yを距離空間, x_0をXの要素とする. また, $f : X \backslash \{x_0\} \to Y$とする. このとき, $x \to x_0$のときの写像fの極限は一意に定まることを証明せよ.

3. Xを距離空間, x_0をXの要素とする. また, $f, g, h : X \backslash \{x_0\} \to \mathbb{R}$で,

$$f(x) \leq g(x) \leq h(x) \quad \forall x \in X \backslash \{x_0\}$$

とする. このとき, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = a$ならば, $\lim_{x \to x_0} g(x) = a$が言える. このことを証明せよ.

4. Xを距離空間, x_0をXの要素とする. また, $f, g : X \backslash \{x_0\} \to \mathbb{R}$で,

$$f(x) \leq g(x) \quad \forall x \in X \backslash \{x_0\}$$

とする. このとき, $\lim_{x \to x_0} f(x) = \alpha$, $\lim_{x \to x_0} g(x) = \beta$ならば, $\alpha \leq \beta$となる. このことを証明せよ.

5. Xを距離空間, x_0をXの要素とする. また, $f : X \backslash \{x_0\} \to \mathbb{R}$で, $\lim_{x \to x_0} f(x) = a$とする. このとき, ある正の数$\delta, M > 0$が存在し,

$$0 < d(x, x_0) < \delta \Rightarrow |f(x)| \leq M$$

となる (つまり, x_0のある近傍においては関数fは有界になる). このことを示せ.

6. Xを距離空間, x_0をXの要素とする. また, $f, g : X \backslash \{x_0\} \to \mathbb{R}$, $\lim_{x \to x_0} f(x) = a$, $\lim_{x \to x_0} g(x) = b$とする. このとき,

$$(1) \quad |f(x)| \to |a|$$
$$(2) \quad (f + g)(x) \to a + b$$

as $x \to x_0$となる. このことを示せ.

7. X, Yを距離空間, x_0をXの要素とする. また, $f : X \backslash \{x_0\} \to Y$とする. このとき, 次の3条件

(1) fはx_0において連続である.

(2) $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$

(3) $x \to x_0 \Rightarrow f(x) \to f(x_0)$

i.e. $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $0 < d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$

は同値である. (1)と(2)の同値性については, すでに証明済みである. ここでは, (2)と(3)が同値であることを示せ.

8. $X = (a, \infty) (\subset \mathbb{R})$, Yを距離空間で, $f : X \to Y$とする. ここで, $a \in \mathbb{R}$である. 写像の極限 $f(x) \to y_0$ (as $x \to \infty$) の定義を述べ, 例を挙げて説明せよ.

9. 問題8と同じ設定で, $x \to \infty$のときの写像fの極限は一意に定まることを証明せよ.

10. 問題8と同じ設定で，

$$f(x) \le g(x) \le h(x) \quad \forall x \in X$$

を仮定する．このとき，$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = \alpha$ならば，$\lim_{x\to\infty} g(x) = \alpha$が言える．このことを証明せよ．

11. $X = (-\infty, b)(\subset \mathbb{R})$，$Y$を距離空間で，$f : X \to Y$とする．ここで，$b \in \mathbb{R}$である．関数の極限$f(x) \to \alpha\ (x \to -\infty)$はどう定義すればよいだろうか？

12. 本文中で未証明の事柄で，当然成り立つと思われる結果をピックアップしてそれに証明を与えよ．(うまく証明できれば，あなたは自ら新しい定理を発見，証明できたことになる！)