Continuous mappings (2)

Review

X, Y MSs
$f : X \to Y$
$x_0 \in X$
$\Rightarrow$ Equivalent

① f: continuous at x_0

i.e. $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

② $\forall \varepsilon > 0, \exists \delta > 0$:

$\underbrace{d(x, x_0) < \delta}_{\text{i.e. } x \in S_\delta(x_0)} \Rightarrow \underbrace{d\big(f(x), f(x_0)\big) < \varepsilon}_{\text{i.e. } f(x) \in S_\varepsilon(f(x_0))}$

$\because d(x, x_0) < \delta \Rightarrow d\big(f(x), f(x_0)\big) < \varepsilon$

$\Leftrightarrow f\big(S_\delta(x_0)\big) \subset S_\varepsilon\big(f(x_0)\big)$.

Def

X : MS

- $\mathbb{G}_X = \{G \subset X \mid G \text{ is open in } X.\}$
- $\mathbb{F}_X = \{F \subset X \mid F \text{ is closed in } X.\}$
- $\mathbb{B}_X = \{S_\varepsilon(x) \subset X \mid x \in X,\ \varepsilon > 0\}$

※ $\mathbb{B}_X \subset \mathbb{G}_X$

Th

X, Y MSs

$f: X \to Y$

$\Rightarrow$ Equivalent

① f: continuous

i.e. $^\forall x_0 \in X,\ x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

② $^\forall x_0 \in X,\ \varepsilon > 0,\ ^\exists \delta > 0$:

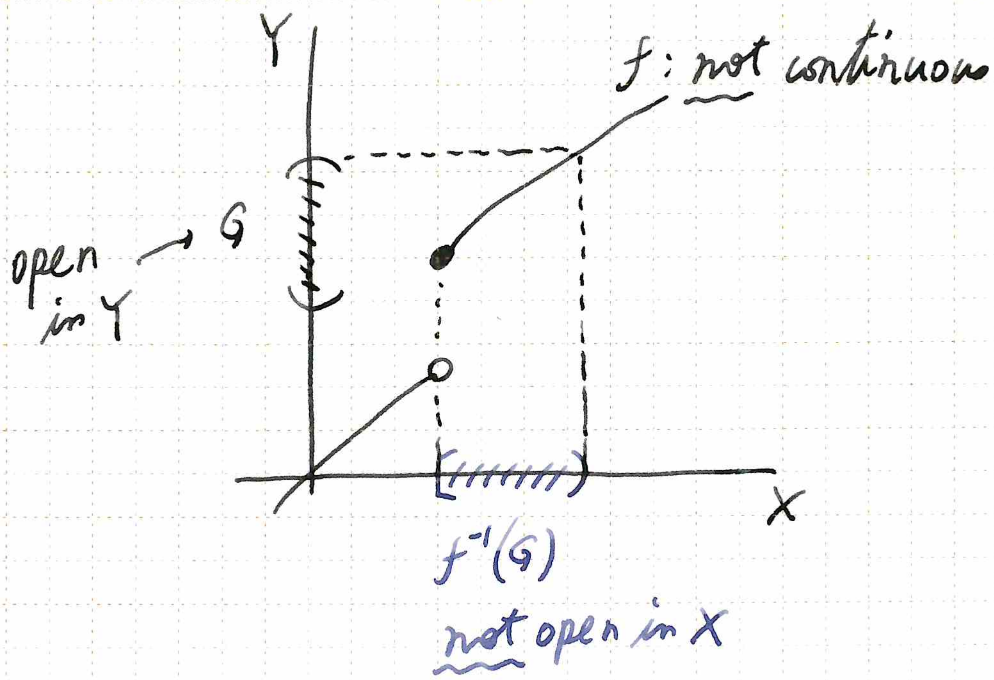
$d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$

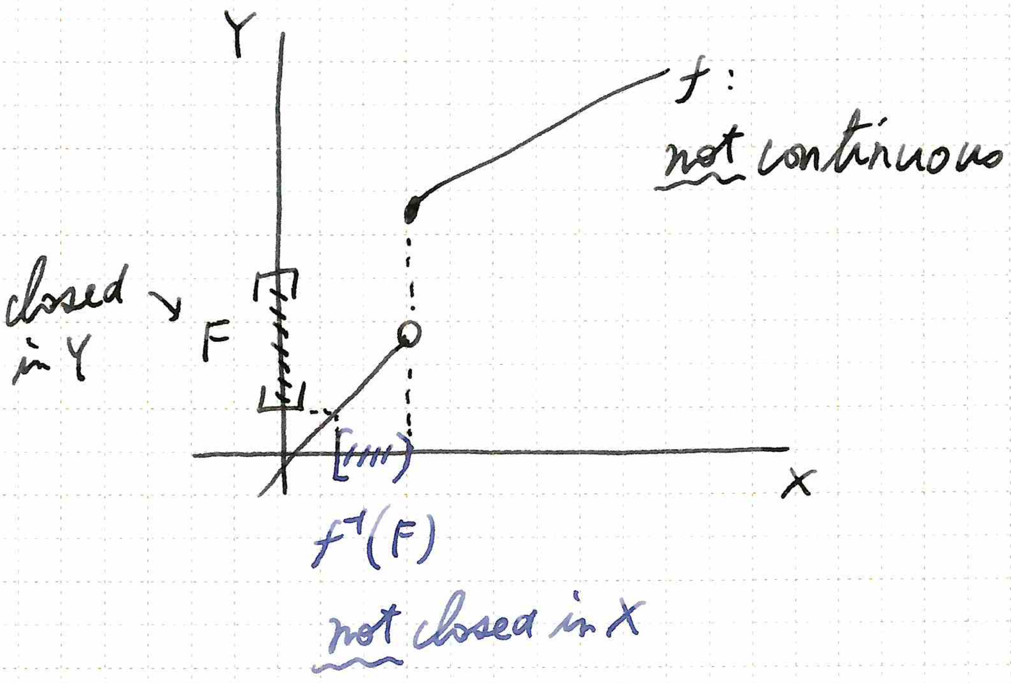
i.e. $f(S_\delta(x_0)) \subset S_\varepsilon(f(x_0))$

③ $^\forall F \in \mathbb{F}_Y,\ f^{-1}(F) \in \mathbb{F}_X$

④ $^\forall G \in \mathbb{G}_Y,\ f^{-1}(G) \in \mathbb{G}_X$

⑤ $^\forall B \in \mathbb{B}_Y,\ f^{-1}(B) \in \mathbb{G}_X$


Y
f : not continuous
open in Y
G
X
f⁻¹(G)
not open in X


Y
f : not continuous
closed in Y
F
X
f⁻¹(F)
not closed in X

① $f: X \to Y$ continuous.

i.e. $\forall x_0 \in X,\ x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

② $\forall x_0 \in X,\ \varepsilon > 0,\ \exists \delta > 0:$

$$d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$$

i.e $f(S_\delta(x_0)) \subset S_\varepsilon(f(x_0))$

③ $\forall F \in \mathbb{F}_Y,\ f^{-1}(F) \in \mathbb{F}_X$

④ $\forall G \in \mathbb{G}_Y,\ f^{-1}(G) \in \mathbb{G}_X$

⑤ $\forall B \in \mathbb{B}_Y,\ f^{-1}(B) \in \mathbb{G}_X$

① $\Leftrightarrow$ ② OK.

We prove that

① $\Rightarrow$ ③

③ $\Rightarrow$ ④

④ $\Rightarrow$ ⑤

⑤ $\Rightarrow$ ②

① ⇒ ③

Let $F \in \mathbb{F}_Y$.

We show that $f^{-1}(F) \in \mathbb{F}_X$.

Let $\{x_n\} \subset f^{-1}(F) \ (\subset X) : x_n \to x \in X$.

i.e. $\forall n \in \mathbb{N},\ f(x_n) \in F$

Our aim is to prove that $x \in f^{-1}(F)$.

i.e $f(x) \in F$

As $x_n \to x$, we have from ① that

$f(x_n) \to f(x)$.

As $\{f(x_n)\} \subset F$, $f(x_n) \to f(x)$, and $F \in \mathbb{F}_Y$,

we obtain $f(x) \in F$. ⌟

③ ⇒ ④

Let $G \in \mathbb{G}_Y$.

We show that $f^{-1}(G) \in \mathbb{G}_X$.

As $G \in \mathbb{G}_Y$, it follows that $G^c \in \mathbb{F}_Y$.

From ③, $f^{-1}(G^c) \in \mathbb{F}_X$.

Hence, $(f^{-1}(G))^c = f^{-1}(G^c) \in \mathbb{F}_X$.

This means that $f^{-1}(G) \in \mathbb{G}_X$. ⌟

④ ⇒ ⑤ is obvious. ⌟

② $\forall x_0 \in X, \varepsilon > 0, \exists \delta > 0:$

$$f(S_\delta(x_0)) \subset S_\varepsilon(f(x_0))$$

⑤ $\forall B \in \mathbb{B}_Y, f^{-1}(B) \in \mathbb{G}_X$

⑤ ⇒ ②

Let $x_0 \in X$ and $\varepsilon > 0$.

Then, $f(x_0) \in Y$ and $S_\varepsilon(f(x_0)) \in \mathbb{B}_Y$.

Define $G \equiv f^{-1}(S_\varepsilon(f(x_0))) \ (\subset X)$.

Then, the following hold:

(1) $\underline{G \in \mathbb{G}_X}$ From (5), OK.

(2) $\underline{x_0 \in G} = f^{-1}(S_\varepsilon(f(x_0)))$

i.e. $f(x_0) \in S_\varepsilon(f(x_0))$. OK.

(3) $\underline{f(G) \subset S_\varepsilon(f(x_0))}$

Indeed, $f(G) = f(f^{-1}(\underline{S_\varepsilon(f(x_0))}))$

$\subset \underline{S_\varepsilon(f(x_0))}$. ⌟

From (1) and (2), $\exists \delta > 0 : S_\delta(x_0) \subset G$.

Using (3), we obtain

$$f(S_\delta(x_0)) \subset f(G) \subset S_\varepsilon(f(x_0)).$$

//

$$\boxed{\begin{array}{l} ② \ {}^\forall x_0 \in X, \varepsilon > 0, {}^\exists \delta > 0 : \\ \qquad f(S_\delta(x_0)) \subset S_\varepsilon(f(x_0)) \\ ④ \ {}^\forall G \in \mathcal{G}_Y, f^{-1}(G) \in \mathcal{G}_X \end{array}}$$

<u>② ⇒ ④</u> (alternative proof)

Let $G \in \mathcal{G}_Y$.

We prove that <u>$f^{-1}(G) \in \mathcal{G}_X$.</u>

i.e. ${}^\forall x \in f^{-1}(G), {}^\exists \delta > 0 : S_\delta(x) \subset f^{-1}(G)$.

Let $x \in f^{-1}(G)$ $(\subset X)$.

Then, $f(x) \in G$.

As $G \in \mathcal{G}_Y$, ${}^\exists \varepsilon > 0 : S_\varepsilon(f(x)) \subset G$. — (*)

From ②, for $x \in X$ and $\varepsilon > 0$,

${}^\exists \delta > 0 : f(S_\delta(x)) \subset S_\varepsilon(f(x))$. — (**)

From (*) and (**),

$f(S_\delta(x)) \subset G$.

Therefore, $f^{-1}(f(S_\delta(x))) \subset f^{-1}(G)$.

We have

$S_\delta(x) \subset f^{-1}(f(S_\delta(x))) \subset f^{-1}(G)$.

$\therefore {}^\forall x \in f^{-1}(G), {}^\exists \delta > 0 : S_\delta(x) \subset f^{-1}(G)$. //

① f: continuous

i.e. $\forall x_0 \in X$, $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

④ $\forall G \in \mathcal{G}_Y$, $f^{-1}(G) \in \mathcal{G}_X$

①⇒④ (alternative proof)

Let $G \in \mathcal{G}_Y$.

We prove that $f^{-1}(G) \in \mathcal{G}_X$.

i.e. $\forall x \in f^{-1}(G)$, $\{x_n\} \subset (f^{-1}(G))^c$,

$x_n \not\to x$

Let $x \in f^{-1}(G)$ and $\{x_n\} \subset (f^{-1}(G))^c = f^{-1}(G^c)$.

i.e. $f(x) \in G$

As $\{x_n\} \subset f^{-1}(G^c)$,

we have that $\{f(x_n)\} \subset G^c$.

As $G \in \mathcal{G}_Y$, $f(x) \in G$, and $\{f(x_n)\} \subset G^c$,

we obtain $f(x_n) \not\to f(x)$.

From ①, $x_n \not\to x$.

This implies that $f^{-1}(G) \in \mathcal{G}_X$. //

Review

Th

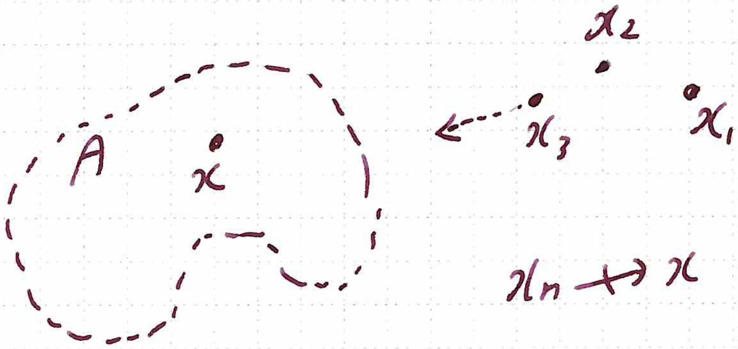
(X, d) MS

$A \subset X$

$\Rightarrow$ Equivalent

① A : open in X.

② $\forall x \in A$, $\{x_n\} \subset A^c$, $x_n \not\to x$



$x_n \not\to x$

$$\boxed{\begin{array}{l} X \ \text{MS} \\ f: X \to \mathbb{R} \text{ continuous} \\ a \in \mathbb{R} \\ A = \{ y \in \mathbb{R} \mid y \geqq a \} \\ \Rightarrow f^{-1}(A) \ (\subset X) \text{ closed} \end{array}}$$

← special case

Proof

Let $\{x_n\} \subset f^{-1}(A)$: $x_n \to x \in X$.

i.e. $\forall n \in \mathbb{N}$, $x_n \in f^{-1}(A)$

i.e. $\forall n \in \mathbb{N}$, $f(x_n) \in A$

i.e. $\forall n \in \mathbb{N}$, $f(x_n) \geqq a$ —(*)

We demonstrate that $x \in f^{-1}(A)$.

i.e. $f(x) \in A$

i.e. $f(x) \geqq a$.

As $x_n \to x$ and f is continuous,

we have $f(x_n) \to f(x)$. —(**)

From (*) and (**), we obtain $f(x) \geqq a$.

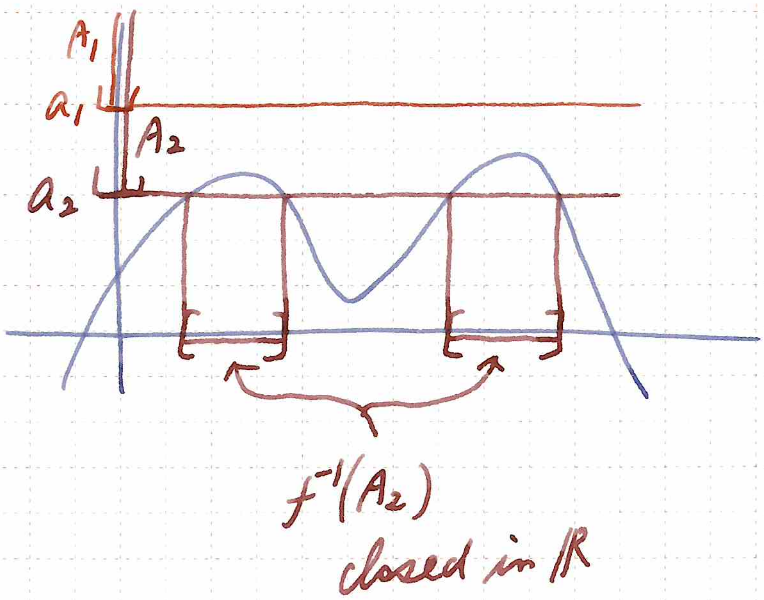
//

ex

$f: \mathbb{R} \to \mathbb{R}$

$A_1 = \{ y \in \mathbb{R} \mid y \geqq a_1 \}$ closed in $\mathbb{R}$

$A_2 = \{ y \in \mathbb{R} \mid y \geqq a_2 \}$ closed in $\mathbb{R}$



$f^{-1}(A_1) = \{ x \in \mathbb{R} \mid f(x) \geqq a_1 \} = \emptyset$

closed in $\mathbb{R}$.

ex

$X = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f: X \to \mathbb{R}$ defined by

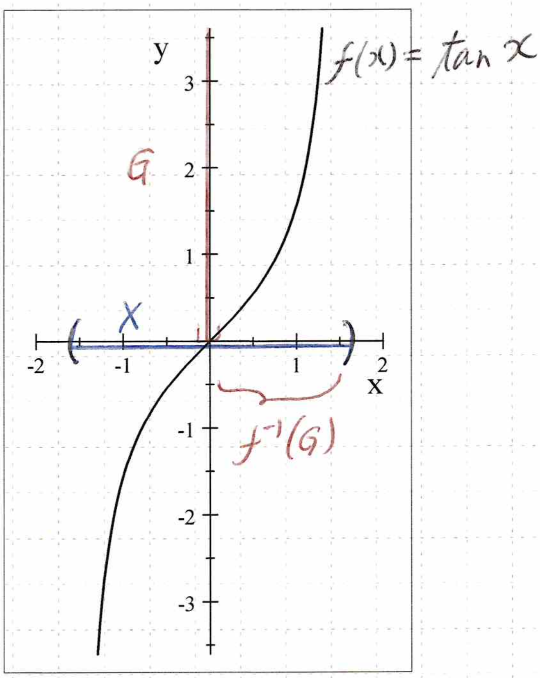
$f(x) = \tan x \quad \forall x \in X$ ← f: continuous

$G = [0, \infty)$ closed in $\mathbb{R}$.

Then, $f^{-1}(G) = \left[0, \frac{\pi}{2}\right)$ is not closed in $\mathbb{R}$.

However,

$f^{-1}(G) = \left[0, \frac{\pi}{2}\right)$ is closed in X.



ex

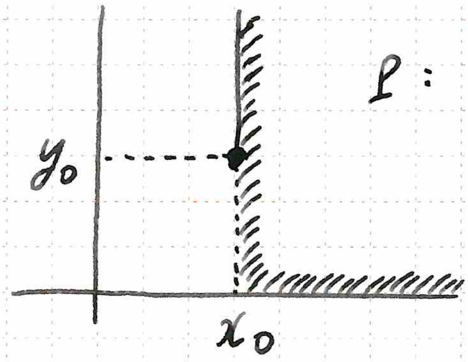
goods X, Y

$(x_0, y_0) \precsim (x, y)$, where $(x_0, y_0), (x, y) \in \mathbb{R}^2_+$.

$\iff \begin{cases} ① \ x_0 < x \ \text{ or} \\ ② \ x_0 = x \text{ and } y_0 \leq y \end{cases}$

<lexicographical order> 辞書的順序

$P = \{(x, y) \in \mathbb{R}^2_+ \mid (x_0, y_0) \precsim (x, y)\}$

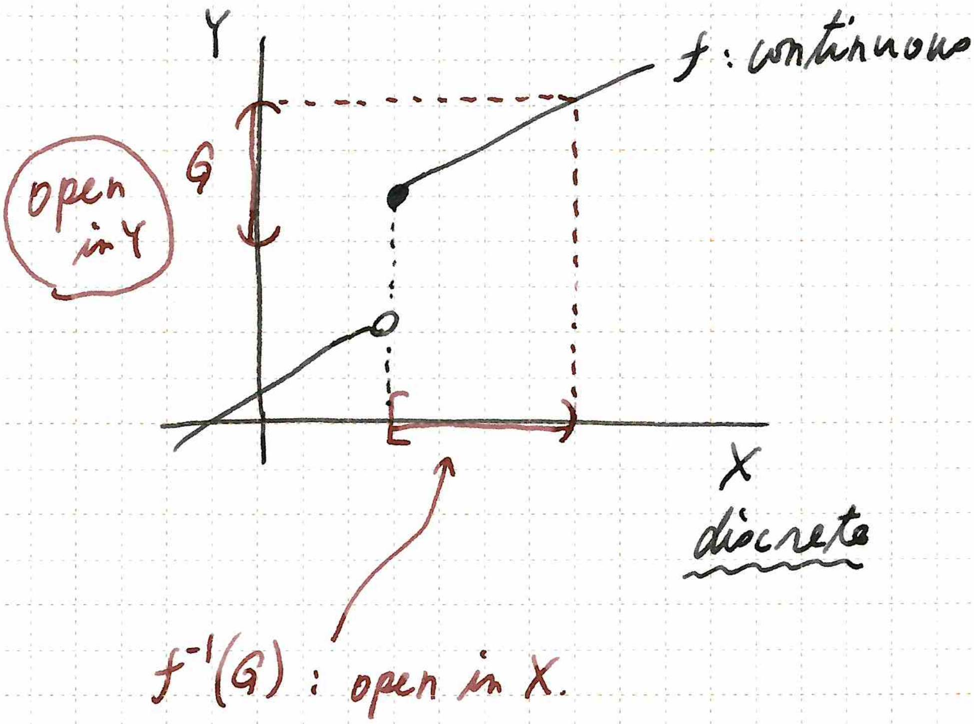


P: (x_0, y_0)と同等以上に好まれる消費パターンの集合

ある意味で"不連続な好み"

X <u>discrete</u> MS
Y MS
$f : X \to Y$
$\Rightarrow f$ is continuous

i.e. $\forall G \in \mathcal{G}_Y,\ f^{-1}(G) \in \mathcal{G}_X$.

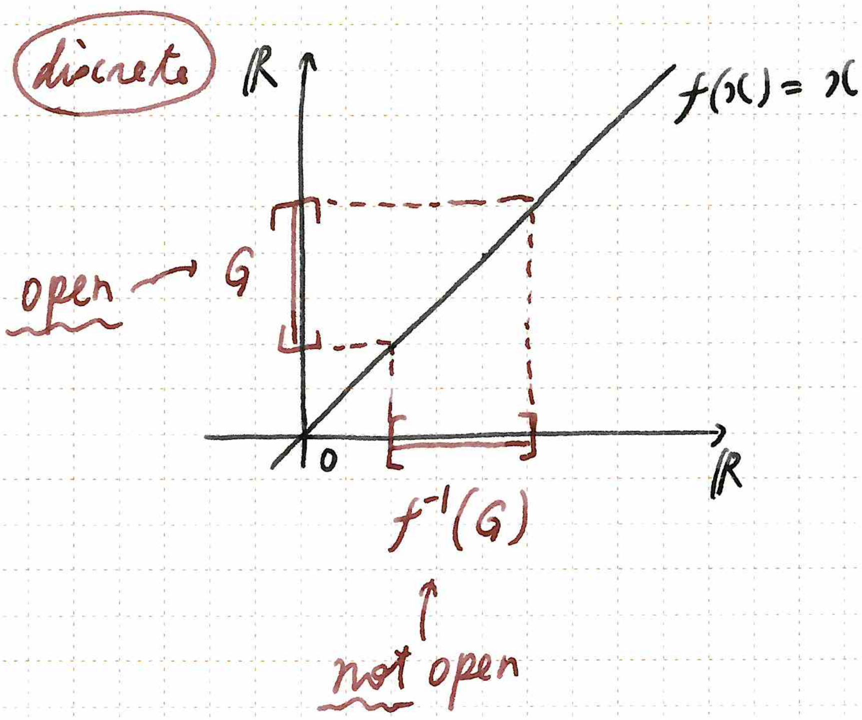


ex

$f(x) = x$

$f : \mathbb{R} \to \mathbb{R}$ not continuous

discrete metric



※ X に open set が多いほど f は連続になりやすい。
Y に 〃 〃 , f は 〃 なりにくい。

Next aim:

the intermediate value theorem (Appendix)

Th

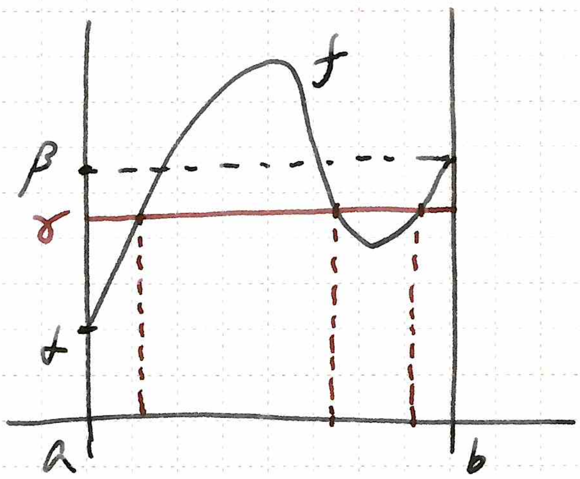
$f : [a,b] \to \mathbb{R}$ continuous

$\alpha = f(a)$

$\beta = f(b)$

$\Rightarrow \forall \gamma \in (\alpha, \beta)$ (or (β, α)),

$\exists c \in (a,b) : f(c) = \gamma$



Lemma

X MS

$f: X \to \mathbb{R}$ continuous

$x_0 \in X : f(x_0) > 0$

$\Rightarrow \exists \delta > 0 : d(x, x_0) < \delta \Rightarrow f(x) > 0$

Proof.

Let $\varepsilon = f(x_0) > 0$.

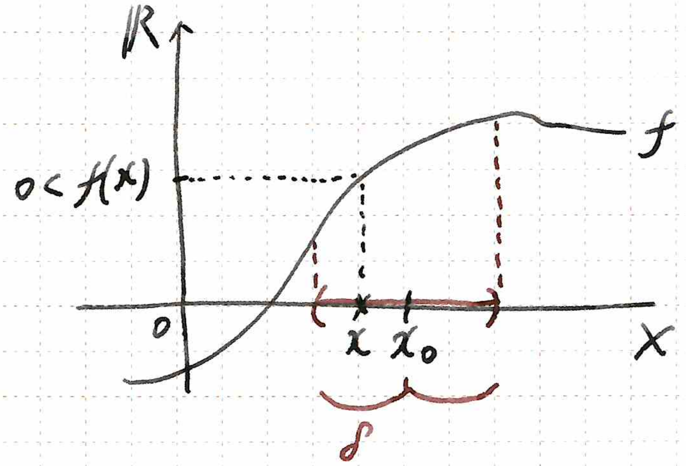
As f is continuous, for $\varepsilon > 0$,

$\exists \delta > 0 : d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

$$\therefore \underbrace{f(x_0) - \varepsilon}_{=0} < f(x) < f(x_0) + \varepsilon$$

$\therefore \exists \delta > 0 : d(x, x_0) < \delta \Rightarrow f(x) > 0$.

//



$$\boxed{\begin{array}{l} A \subset [a, b] \subset \mathbb{R} \\ A \neq \emptyset \\ \Rightarrow \alpha \equiv \sup A \in [a, b] \end{array}}$$

<u>Proof</u>

As A is $\neq \emptyset$ and bdd above,

$\alpha \equiv \sup A \in \mathbb{R}$ exists.

That is,
$$\begin{cases} ① \ \forall x \in A, \ x \leqq \alpha, \\ ② \ \forall x \in A, \ x \leqq \beta \Rightarrow \alpha \leqq \beta. \end{cases}$$

We show that $\underline{a \leqq \alpha \leqq b}$.

<u>$a \leqq \alpha$</u>

As $A \subset [a, b]$, we have that

$\forall x \in A, \ a \leqq x$.

From ①, $a \leqq \alpha$. ⌟

<u>$\alpha \leqq \beta$</u>

As $A \subset [a, b]$, $\forall x \in A, \ x \leqq b$.

From ②, $\alpha \leqq b$.

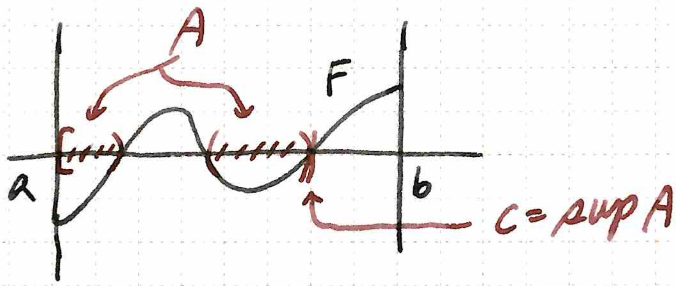
//

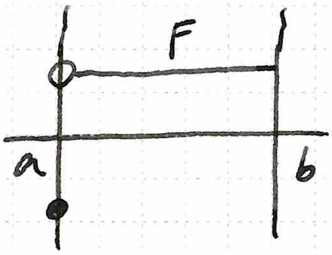
$$\boxed{\begin{array}{l} F:[a,b]\to\mathbb{R} \\ F(a)<0 \\ A\equiv\{x\in[a,b]\mid F(x)<0\} \\ \Rightarrow \exists c\equiv\sup A\in[a,b] \end{array}}$$

Proof

As $a\in A$, $A\neq\emptyset$.

Therefore, $\exists c\equiv\sup A\in[a,b]$. //

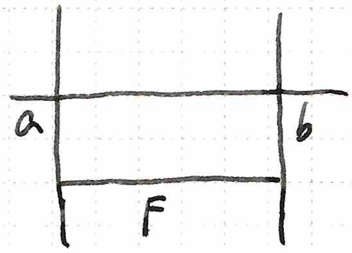




← In this case,

$A=\{a\}$.

$\therefore \sup A = a\in[a,b]$



← In this case,

$A=[a,b]$.

$\therefore \sup A = b$.

Th

$$\boxed{\begin{array}{l} f:[a,b]\to\mathbb{R} \text{ continuous} \\ \alpha \equiv f(a) < \beta \equiv f(b) \\ \Longrightarrow \forall \gamma \in (\alpha,\beta),\ \exists c\in(a,b): f(c)=\gamma \end{array}}$$

Proof

Define $F(x) = f(x) - \gamma$.

We show that $\exists c \in (a,b): F(c) = 0$.

It follows that

- $F:[a,b]\to\mathbb{R}$ is continuous,
- $F(a) = f(a) - \gamma = \alpha - \gamma < 0$,
- $F(b) = f(b) - \gamma = \beta - \gamma > 0$.

Define $A \equiv \{x \in [a,b] \mid F(x) < 0\}$ $(\subset [a,b])$.

As $a \in A$, $A \neq \emptyset$.

We have that $\exists c \equiv \sup A \in [a,b]$.

$F(c) \leqq 0$ — ①

As $c \equiv \sup A$, $\exists \{x_n\} \subset A : x_n \to c$.

As $x_n \in A$, $F(x_n) < 0$ $(\forall n \in \mathbb{N})$.

As F is continuous and $x_n \to c$,

$$F(c) = F(\lim x_n) = \lim F(x_n) \leqq 0.$$

As $F(b)>0$ and F is continuous,

$\exists d \in (a,b) : d < x \leqq b \Rightarrow F(x) > 0.$

$c \leqq d \ (<b)$ — ②

If $c > d$, we have $F(c) > 0$.

This contradicts ①. ⌟

Define $B \equiv (c, b]$.

From ②, $B \neq \phi$.

Let $\{x_n\} \subset B : x_n \to c$.

$\left(\text{e.g., let } x_n = \frac{1}{n} b + \left(1 - \frac{1}{n}\right) c.\right)$

As $x_n \in B = (c, b]$, $F(x_n) \geqq 0$.

As F is continuous and $x_n \to c$,

$F(c) = F(\lim x_n) = \lim F(x_n) \geqq 0.$

This together with ① shows that

$F(c) = 0.$

As $F(a) < 0$, we obtain from ② that

$c \in (a,b)$.

$\therefore \forall \gamma \in (\alpha, \beta), \exists c \in (a,b) : F(c) = 0.$ //

Cor

$f : [a, b] \to \mathbb{R}$ continuous

$\alpha = f(a)$, $\beta = f(b)$

$\Rightarrow \forall \gamma \in [\alpha, \beta]$ (or $[\beta, \alpha]$),

$\exists c \in [a, b] : f(c) = \gamma$

Cor

$f : [a, b] \to \mathbb{R}$ continuous

$\alpha = f(a)$, $\beta = f(b)$: $\alpha < \beta$

$\Rightarrow \forall \gamma \in [\alpha, \beta)$,

$\exists c \in [a, b) : f(c) = \gamma$

Continuous mappings (2)

1. X, Yを距離空間, $f : X \to Y$とする. 次の5条件が同値である. ((1)と(2)が同値であることは, 前講で証明済みである.)

(1) fはX上の連続関数である.

(2) $\forall x_0 \in X,\ \varepsilon > 0,\ \exists \delta > 0$ s.t. $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$

(3) Yの任意の閉集合Fについて, $f^{-1}(F)$がXの閉集合になる.

(4) Yの任意の開集合Gについて, $f^{-1}(G)$がXの開集合になる.

(5) Yの任意の開球Bについて, $f^{-1}(B)$がXの開集合になる.

このことを, (1) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (2)を順に示すことで証明せよ. また, なぜこれだけで(1) – (5)のすべてが互いに同値であることを証明できたことになるのか説明せよ.

2. 問題1の(5)について, fが連続であっても, 開球の逆像は開球になるとは限らないが, 開集合にはなる. このことを$X = Y = \mathbb{R}$の場合について, fのグラフを書いて納得せよ.

3. 問題1の(1) – (5)について, 問題1で証明した(1) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (2)以外の部分についても, 証明を考えてみよ.

4. Xを距離空間, $f : X \to \mathbb{R}$を連続関数とする. 実数bについて, $f^{-1}((-\infty, b])$が閉集合であることを示せ.

5. X, Yを距離空間, $f : X \to Y$とする. 定義域Xが離散距離空間だとすると, fは必ず連続になる. なぜか?

6. X, Yを距離空間, $f : X \to Y$とする. 値域Yが離散距離空間だとすると, fは連続になりにくい. なぜか? また, その場合でもfが連続になるのは, 例えばどのような場合か?

以下, **おまけ**

7. Xを距離空間, $f : X \to \mathbb{R}$を連続関数とする. 定義域の点$x_0 \in X$が$f(x_0) > 0$を満たすとすると, x_0のある近傍が存在して, その任意の点xについても$f(x) > 0$となる. このことを示せ.

8. $A \subset [a, b] \subset \mathbb{R}$で$A \neq \emptyset$を仮定する. このとき, 実数$\alpha \equiv \sup A$が存在し, $a \leq \alpha \leq b$となることを示せ.

9. 関数$f : \mathbb{R} \to \mathbb{R}$について, 中間値の定理を証明せよ.