Limits (1)

$\{a_n\} \subset \mathbb{R}$
$a \in \mathbb{R}$

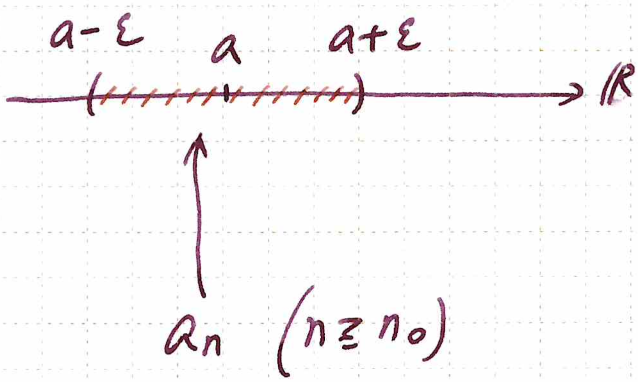
- <u>$\{a_n\}$ converges to a.</u>

$\Leftrightarrow \forall_{\varepsilon>0}, \exists n_0 \in \mathbb{N}:$

$$n \geq n_0 \Rightarrow |a_n - a| < \varepsilon.$$

$$\Leftrightarrow -\varepsilon < a_n - a < \varepsilon$$

$$\Leftrightarrow a - \varepsilon < a_n < a + \varepsilon$$



$\Leftrightarrow a_n \to a \ (n \to \infty)$

ex

$$\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\} (\subset \mathbb{R})$$
converges to $0 \; (\in \mathbb{R})$.

Proof.

Let $\varepsilon > 0$.

Choose $n_0 \in \mathbb{N}$: $\frac{1}{\varepsilon} < n_0$.

Then, $\frac{1}{n_0} < \varepsilon$.

We have that

$$n \geqq n_0 \Rightarrow \frac{1}{n} \leqq \frac{1}{n_0} < \varepsilon.$$

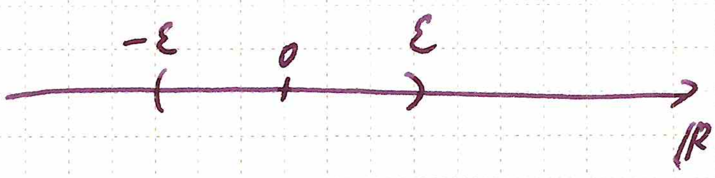
$$\therefore \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow -\varepsilon < 0 < \frac{1}{n} < \varepsilon.$$

This shows that $a_n \to 0$. //

ex

$\{0, 0, 0, \cdots\}$ converges to $0 \; (\in \mathbb{R})$.

i.e. $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow -\varepsilon < \underset{0}{\underset{\shortparallel}{a_n}} < \varepsilon$



$a_n \to a$

$\Leftrightarrow {}^{\forall}\varepsilon > 0,\ {}^{\exists}n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow |a_n - a| < \varepsilon$

$\Big\downarrow$ negation

$a_n \not\to a$

$\Leftrightarrow {}^{\exists}\varepsilon > 0 : {}^{\forall}n \in \mathbb{N},\ {}^{\exists}n' \geqq n : |a_{n'} - a| \geqq \varepsilon$

ex

$\boxed{\begin{array}{l} \{a_n\} = \{1, 0, 1, 0, \cdots\} \\ \quad \text{does not converge to } 0 \ (\in \mathbb{R}). \end{array}}$

Proof

Let $\varepsilon = \frac{1}{2} > 0$.

Let $n \in \mathbb{N}$.

Clearly, ${}^{\exists}n' \geqq n : a_{n'} = 1$.

Thus, $\underbrace{|a_{n'} - 0|}_{= 1} \geqq \varepsilon = \frac{1}{2}$.

$\therefore {}^{\exists}\varepsilon = \frac{1}{2} > 0 : {}^{\forall}n \in \mathbb{N},\ {}^{\exists}n' \geqq n : |a_{n'} - 0| \geqq \varepsilon$.

This means that $a_n \not\to 0$.

//

- $\{a_n\}$ is convergent.

$\iff \exists a \in \mathbb{R} : a_n \to a$

$\iff \exists a \in \mathbb{R} : \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}:$
$n \geq n_0 \Rightarrow |a_n - a| < \varepsilon.$

- $\{a_n\}$ is not convergent

$\iff \forall a \in \mathbb{R}, \exists \varepsilon > 0 : \forall n \in \mathbb{N},$
$\exists n' \geq n : |a_{n'} - a| \geq \varepsilon$

ex

$\{1, 2, 3, 4, \cdots\}$

$\{1, 0, 1, 0, \cdots\}$

$\{a_n\} \subset \mathbb{R}$

$a \in \mathbb{R}$

$\Rightarrow$ Equivalent

① $a_n \to a$

② $a_n - a \to 0$

③ $|a_n - a| \to 0$

$$\boxed{\begin{array}{l} \{a_n\} \subset \mathbb{R} \\ a_n \to a \\ a_n \to b \\ \Longrightarrow a = b \end{array}}$$

極限の一意性

<u>Proof.</u>

Suppose for the sake of contradiction that

$a \neq b$.

Then, $|a-b| > 0$.

Let $\varepsilon = \frac{1}{3}|a-b| > 0$.

As $a_n \to a$,

$\exists n_1 \in \mathbb{N} : n \geqq n_1 \Rightarrow |a_n - a| < \varepsilon$.

As $a_n \to b$,

$\exists n_2 \in \mathbb{N} : n \geqq n_2 \Rightarrow |a_n - b| < \varepsilon$.

Let $n_0 = \max\{n_1, n_2\} \in \mathbb{N}$.

Then, $|a-b| \leqq |a - a_{n_0}| + |a_{n_0} - b|$

$< 2\varepsilon < 3\varepsilon = |a-b|$.

This is a contradiction.

//

※ $a_n \to a \iff a = \lim_{n \to \infty} a_n$

Def.

$\{a_n\} \subset \mathbb{R}$ <u>monotone increasing</u>

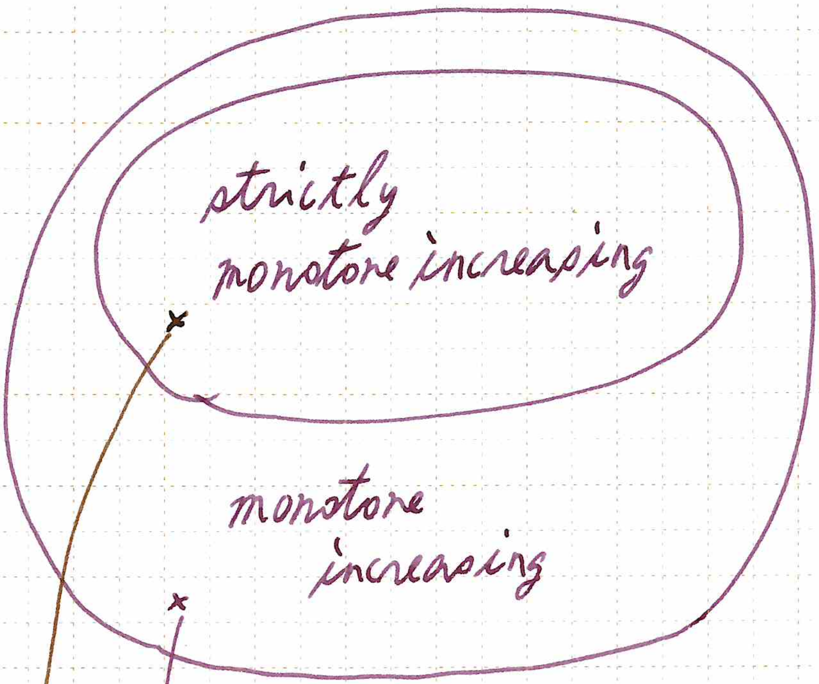
$\Longleftrightarrow m \leqq n \Longrightarrow a_m \leqq a_n$

Def.

$\{a_n\} \subset \mathbb{R}$ <u>strictly</u>

<u>monotone increasing</u>

$\Longleftrightarrow m < n \Longrightarrow a_m < a_n$



- $\{1, 1, 2, 3, 4, 5, 5, 6, \cdots\}$
- $\{1, 1, 1, 1, \cdots\cdots\}$

- $\{1, 2, 3, 4, \cdots\cdots\}$
- $\{3, 3.1, 3.14, 3.141, \cdots\cdots\}$

Lemma

$A \subset \mathbb{R}$, $\neq \emptyset$

$\alpha = \sup A \in \mathbb{R}$

$\Rightarrow \forall_{\varepsilon > 0}, \exists_{x \in A} : \alpha - \varepsilon < x$

Proof

Suppose by contradiction that

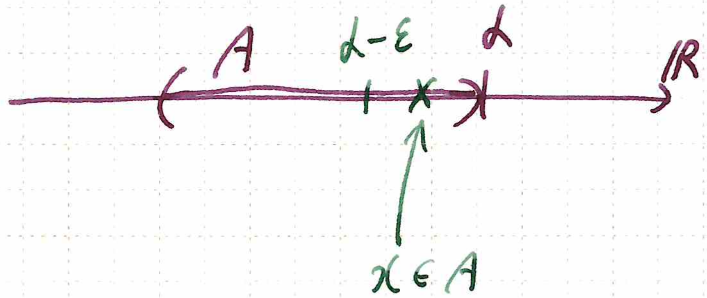
$\exists_{\varepsilon > 0} : \forall_{x \in A}, \ \alpha - \varepsilon \geqq x$.

This shows that $\alpha - \varepsilon$ is an upper bound of A.

As $\alpha = \sup A$, $\alpha \leqq \alpha - \varepsilon$.

This is a contradiction.

//



Th

> $\{a_n\} \subset \mathbb{R}$ bdd above,
> monotone increasing
>
> $\Rightarrow a_n \to \alpha \equiv \sup_k a_k$

Proof

As $\{a_n\} (\subset \mathbb{R})$ is bdd above, $\exists \alpha \equiv \sup a_n \in \mathbb{R}$.

$\therefore \begin{cases} ① \ \forall n \in \mathbb{N},\ a_n \leqq \alpha, \\ ② \ \forall n \in \mathbb{N},\ a_n \leqq a \Rightarrow \alpha \leqq a. \end{cases}$

We show that

$$\underline{\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow \alpha - \varepsilon < a_n < \alpha + \varepsilon.}$$

Let $\varepsilon > 0$.

From Lemma, $\exists n_0 \in \mathbb{N}: \alpha - \varepsilon < a_{n_0}$.

Let $n \geqq n_0$.

As $\{a_n\}$ is monotone increasing, $a_{n_0} \leqq a_n$.

Using ①, we have

$$\alpha - \varepsilon < a_{n_0} \leqq a_n \leqq \alpha < \alpha + \varepsilon.$$

$\therefore \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow \alpha - \varepsilon < a_n < \alpha + \varepsilon$.

This means that $a_n \to \alpha \equiv \sup_k a_k$.

//

<u>ex</u>

$\{a_n\} = \{1, 2, 3, 4, \cdots\}$

- $\{a_n\}$ is <u>not</u> convergent.
- $\{a_n\}$ is monotone increasing.
- $\{a_n\}$ is <u>not</u> bdd above.

<u>ex</u>

$\{b_n\} = \{1, 0, 1, 0, \cdots\}$

- $\{b_n\}$ is <u>not</u> convergent.
- $\{b_n\}$ is <u>not</u> monotone increasing.
- $\{b_n\}$ is bdd above.

- $a_n = \left(1+\frac{1}{n}\right)^n \quad (n \in \mathbb{N})$

Then, $a_1 = \left(1+\frac{1}{1}\right)^1 = 2$

$$a_2 = \left(1+\frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$a_3 = \left(1+\frac{1}{3}\right)^3 = \left(\frac{4}{3}\right)^3$$

$$\vdots$$

Th

$\{a_n\}\ (\subset \mathbb{R})$ is bdd above and monotone increasing.

Cor

$\{a_n\}$ is convergent.

Def.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

ネピアの数

$$\boxed{\begin{array}{l} \{a_n\}, \{b_n\} \subset \mathbb{R} \\ a_n \leq b_n \quad \forall n \in \mathbb{N} \\ a_n \to a \\ b_n \to b \\ \Longrightarrow a \leq b \end{array}}$$

<u>Proof.</u>

Suppose by contradiction that $a > b$.

Let $\varepsilon = \frac{1}{3}(a-b) > 0$.

As $a_n \to a$,

$\exists n_1 \in \mathbb{N} : n \geq n_1 \Rightarrow a - \varepsilon < a_n \ (< a + \varepsilon)$.

As $b_n \to b$,

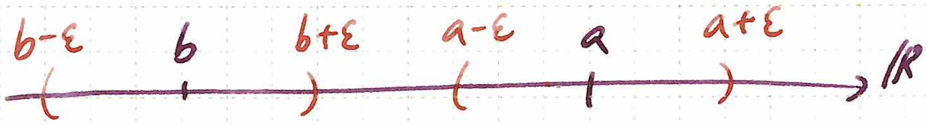
$\exists n_2 \in \mathbb{N} : n \geq n_2 \Rightarrow (b - \varepsilon <)\ b_n < b + \varepsilon$.

Let $n_0 = \max\{n_1, n_2\} \in \mathbb{N}$.

For $n \geq n_0 \ (\geq n_1, n_2)$, we have

$a_n \leq b_n < b + \varepsilon < a - \varepsilon < a_n$.

This is a contradiction. //



$\{a_n\}, \{b_n\} \subset \mathbb{R}$

$a_n \leq b_n \quad {}^\forall n \in \mathbb{N}$

$a_n \to a \quad (a = \lim a_n)$

$b_n \to b \quad (b = \lim b_n)$

$\Rightarrow a \leq b \quad (\lim a_n \leq \lim b_n)$

書き換え

$\{a_n\}, \{b_n\} \subset \mathbb{R}$ convergent

$a_n \leq b_n \quad {}^\forall n \in \mathbb{N}$

$\Rightarrow \lim a_n \leq \lim b_n$

極限をとるという操作は
順序保存的 (order-preserving)
である。

$$\boxed{\begin{array}{l} \{a_n\}, \{b_n\} \subset \mathbb{R} \\ a_n < b_n \quad \forall n \in \mathbb{N} \\ a_n \to a \\ b_n \to b \\ \not\Rightarrow a < b \end{array}}$$

strong

strong

<u>ex</u>

$\{a_n\} = \{0, 0, 0, \cdots\}$

$\{b_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

Then, $a_n < b_n \quad \forall n \in \mathbb{N}$

However, $\underset{\substack{\| \\ 0}}{\lim a_n} \neq \underset{\substack{\| \\ 0}}{\lim b_n}$

$\{a_n\}, \{b_n\}, \{c_n\} \subset \mathbb{R}$

$a_n \leqq b_n \leqq c_n \quad \forall n \in \mathbb{N}$

$a_n \to a$

$c_n \to a$

$\Rightarrow b_n \to a$

はさみうちの原理

<u>Proof.</u>

Let $\varepsilon > 0$.

As $a_n \to a$,

$\exists n_1 \in \mathbb{N} : n \geqq n_1 \Rightarrow a - \varepsilon < a_n \, (< a + \varepsilon)$.

As $c_n \to a$,

$\exists n_2 \in \mathbb{N} : n \geqq n_2 \Rightarrow (a - \varepsilon <) \, c_n < a + \varepsilon$.

Let $n_0 = \max\{n_1, n_2\} \in \mathbb{N}$.

Let $n \geqq n_0$.

Then, $a - \varepsilon < a_n \leqq b_n \leqq c_n < a + \varepsilon$.

$\therefore \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow a - \varepsilon < b_n < a + \varepsilon$.

$\therefore b_n \to a$.

//

Cor

$\{b_n\}, \{c_n\} \subset \mathbb{R}$

$0 \leq b_n \leq c_n \quad \forall n \in \mathbb{N}$

$c_n \to 0$

$\Longrightarrow b_n \to 0$

Limits (1)

1. 数列$\{a_n\}$がある実数$a \in \mathbb{R}$に収束することと収束しないことの定義を述べよ.

2. 数列$\{a_n\} = \left\{\frac{1}{n^2}\right\}$は0に収束するが，任意にとった$\varepsilon(> 0)$に対して$n_0 \in \mathbb{N}$をどのようにとればよいか？

3. 数列$\{a_n\} = \{1, 0, 1, 0, \cdots\}$を考える.
(1) $\{a_n\}$は0に収束しないことを証明せよ.
(2) 同様に考えて，$\{a_n\}$は1に収束しないことを納得せよ(丁寧に証明を書かなくてもよい).
(3) $\{a_n\}$は1/2に収束しないことを証明せよ.
(4) $\{a_n\}$は収束しないことを納得せよ.

4. $\{a_n\}$が収束数列だとすると，極限は一意に定まることを証明せよ.

5. Aを空でない$\mathbb{R}$の部分集合，$\alpha \in \mathbb{R}$をその上限とする. 任意の$\varepsilon > 0$に対して，ある$x \in A$が存在し$\alpha - \varepsilon < x$となることを示せ.

6. 下限についても，問題5と同様なことが言える. どのような命題が成り立つか，書いてみよ.

7. 上に有界な単調増加数列は，その上限に収束する. このことを示せ. また，数列が上に有界であることと単調増加であることは，それぞれ証明にどのように効いているか考えよ.

8. 下に有界な単調減少数列は，その下限に収束する. このことを示せ. また，数列が下に有界であることと単調減少であることは，それぞれ証明にどのように効いているか考えよ.

9. 収束数列について，極限をとるという操作は“順序保存的”であることを示せ.

10. “はさみうちの原理”を証明せよ.

※問題4, 9, 10において，定理のステートメントを学生自ら書き下し，その上でそれを証明できるようになってもらうことが問題の目的である.