Review

(X, d) MS

(d1) $d(x, y) \geqq 0$;

$d(x, y) = 0 \Longleftrightarrow x = y$

(d2) $d(x, y) = d(y, x)$

(d3) $d(x, y) \leqq d(x, z) + d(z, y)$

- $\{x_n\} \subset X$, $x \in X$

  $x_n \to x$

  $\Longleftrightarrow d(x_n, x) \to 0$

  $\Longleftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N};\ n \geqq n_0 \Rightarrow d(x_n, x) < \varepsilon$

- $\{x_n\} \subset X$: <u>convergent</u>

  $\Longleftrightarrow \exists x \in X : x_n \to x$

- $\{x_n\} \subset X$ is <u>not</u> convergent

  $\Longleftrightarrow \forall x \in X,\ x_n \not\to x$

  $\Longleftrightarrow \forall x \in X,\ \exists \varepsilon > 0 : \forall n \in \mathbb{N},$

  $\exists n' \geqq n : d(x_{n'}, x) \geqq \varepsilon$

Th

(X,d) MS

$\{x_n\} \subset X$, $x \in X$

$\Rightarrow$ Equivalent

① $x_n \to x$

② $\forall \{x_{n_i}\} \subset \{x_n\}$, $x_{n_i} \to x$

③ $\forall \{x_{n_i}\} \subset \{x_n\}$,

$\exists \{x_{n_{i_j}}\} \subset \{x_{n_i}\} : x_{n_{i_j}} \to x$

Th

(X,d) MS

$\{x_n\} \subset X$, $x \in X$

$\Rightarrow$ Equivalent

① $x_n \not\to x$

② $\exists \varepsilon > 0$, $\{x_{n_i}\} \subset \{x_n\}$:

$\forall i \in \mathbb{N}$, $d(x_{n_i}, x) \geq \varepsilon$

(X, d) MS
$A \subset X$

- A : open in X

  $\Leftrightarrow {}^{\forall}x \in A, {}^{\exists}r > 0 : S_r(x) \subset A$

- A : closed in X

  $\Leftrightarrow A^c$: open in X.

  $\Leftrightarrow \{x_n\} \subset A : x_n \to x \in X$
  $\Rightarrow x \in A$

$$\boxed{\begin{array}{l}(X, d) \text{ MS} \\ x \in X \\ \Rightarrow \{x\} : \text{closed in } X.\end{array}}$$

Proof (1)

Let $\{x_n\} \subset \{x\} : x_n \to y \in X$.

i.e. ${}^{\forall}n \in \mathbb{N}, x_n = x$

We prove that $y \in \{x\}$.

i.e. $y = x$.

OK.

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$$\boxed{\begin{array}{l} (X, d) \text{ MS} \\ x \in X \\ \Rightarrow \{x\} : \text{closed in } X. \end{array}}$$

Proof (2)

We show that $\{x\}^c$ is open in X.

i.e. $\forall z \in \{x\}^c,\ \exists r > 0 : S_r(z) \subset \{x\}^c$

i.e. $\forall z \in X : z \neq x,$

$\exists r > 0 : S_r(z) \cap \{x\} = \phi$

i.e $\forall z \in X ; z \neq x,$

$\exists r > 0 : x \notin S_r(z)$

i.e $\forall z \in X : z \neq x,\ \exists r > 0 : d(x, z) \geqq r.$

Let $z \in X : z \neq x.$

Define $r \equiv d(x, z) > 0.$

Then, $d(x, z) = r \geqq r.$

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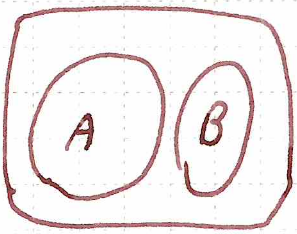
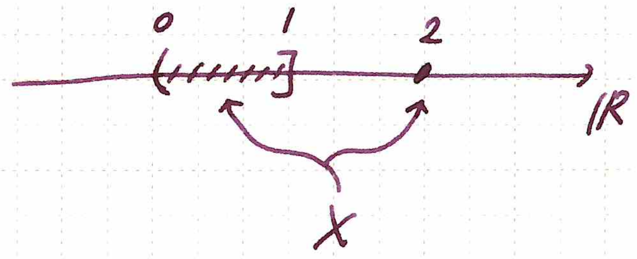
$$\boxed{\begin{array}{l} A \cap B = \phi \\ \Leftrightarrow A \subset B^c \end{array}}$$



ex

$X = (0,1] \cup \{2\}$ subspace of $\mathbb{R}$



Note that

$$S_{\frac{1}{2}}(2) = \{x \in X \mid d(x,2) < \tfrac{1}{2}\}$$
$$= \{2\}.$$

- $\{2\}$: open in X.

For $2 \in \{2\}$, $\exists r = \frac{1}{2} > 0 : S_{\frac{1}{2}}(2) \subset \{2\}$.

- $\{2\}$: closed in X.

OK.

$X, Y \neq \emptyset$

$f : X \to Y$

$A_\mu \subset X$

$B_\mu \subset Y$

Then,

$$A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$$

$$B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$$

$$f^{-1}\left(\bigcup_\mu B_\mu\right) = \bigcup_\mu f^{-1}(B_\mu)$$

$$f^{-1}\left(\bigcap_\mu B_\mu\right) = \bigcap_\mu f^{-1}(B_\mu)$$

$$f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$$

$$f\left(\bigcup_\mu A_\mu\right) = \bigcup_\mu f(A_\mu)$$

$$f\left(\bigcap_\mu A_\mu\right) \subset \bigcap_\mu f(A_\mu)$$

$$f(A^c) \neq (f(A))^c$$

$$A \subset f^{-1}(f(A))$$

$$f(f^{-1}(B)) \subset B$$