Open and closed sets (2)

Review

(X, d) MS

- $G \subset X$: open
  $\iff \forall x \in G, \exists r > 0 : S_r(x) \subset G$

- $F \subset X$ closed
  $\iff F^c$: open in X
  $\iff \{x_n\} \subset F : x_n \to x \in X$
  $\implies x \in F$

- $G \subset X$ open
  $\iff G^c$: closed in X.

- $\{x_n\} \subset X$, $x \in X$
  $\implies$ Equivalent
  ① $x_n \to x$
  ② $\forall \{x_{n_i}\} \subset \{x_n\}$, $x_{n_i} \to x$

$X \neq \emptyset$

$A_\mu \subset X \ (\mu \in M)$

- $x \in \bigcup_{\mu \in M} A_\mu$

  $\Leftrightarrow \exists \mu \in M : x \in A_\mu$

- $x \in \bigcap_{\mu \in M} A_\mu$

  $\Leftrightarrow \forall \mu \in M, x \in A_\mu$

- <u>De Morgan's law</u>

$$\left(\bigcup_\mu A_\mu\right)^c = \bigcap_\mu A_\mu^c$$

$$\left(\bigcap_\mu A_\mu\right)^c = \bigcup_\mu A_\mu^c$$

X metric space

$A \subset C \subset X$

A is closed in X.

$\Rightarrow A$ is closed in C.

Proof

Let $\{x_n\} \subset A : x_n \to x \in C$.

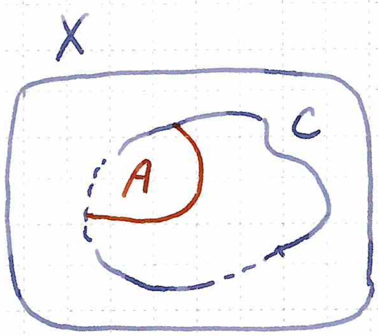
We prove that $\underline{x \in A}$.

As $\{x_n\} \subset A$, $x_n \to x \in (C \subset) X$, and

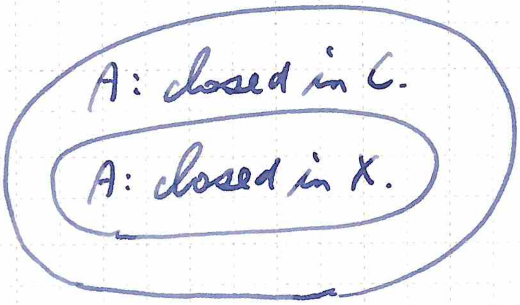
A is closed in X, we obtain $x \in A$.

This means that A is closed in C. //

$\not\Leftarrow$



- A is closed in C.
- A is not closed in X.



X metric space

$C \subset X$ closed

$A \subset C$

$\Rightarrow$ Equivalent

① A : closed in X.

② A : closed in C.

Proof

①$\Rightarrow$② OK.

②$\Rightarrow$①

Let $\{x_n\} \subset A : x_n \to x \in X$.

We show that $x \in A$.

As $\{x_n\} \subset (A \subset) C$, $x_n \to x \in X$, and

C is closed in X, we obtain $x \in C$.

As $\{x_n\} \subset A$, $x_n \to x \in C$, and

A is closed in C, we obtain $x \in A$.

This shows that A is closed in X.

//

Lemma

$A \subset \mathbb{R}$, $\neq \phi$

$\alpha = \sup A \in \mathbb{R}$

$\Rightarrow \exists \{x_n\} \subset A : x_n \to \alpha$

Proof

We know that

$$\forall \varepsilon > 0, \exists x \in A : \alpha - \varepsilon < x \ (\leq \alpha < \alpha + \varepsilon).$$

Letting $\varepsilon = \frac{1}{n} > 0$, we have

$$\forall n \in \mathbb{N}, \exists x_n \in A : \alpha - \frac{1}{n} < x_n \leq \alpha < \alpha + \frac{1}{n}.$$

$\therefore \exists \{x_n\} \subset A : x_n \to \alpha$.

//

点列の話に帰着させる。

Review

$A \subset \mathbb{R}$, $\neq \phi$

$\alpha = \sup A \in \mathbb{R}$

$\Rightarrow \forall \varepsilon > 0, \exists x \in A : \alpha - \varepsilon < x$

$$\boxed{\begin{array}{l} A \subset \mathbb{R} \neq \phi,\ \text{closed},\ \text{bdd above} \\ \Rightarrow \exists \alpha = \sup A = \max A \ (\in A) \end{array}}$$

Proof.

As A is $\neq \phi$ and bdd above,

$\exists \alpha = \sup A \in \mathbb{R}$.

We know that $\exists \{x_n\} \subset A : x_n \to \alpha \in \mathbb{R}$.

As A is closed in $\mathbb{R}$, $\alpha \in A$.

As $\alpha = \sup A \in A$, we have $\alpha = \max A$.

$\therefore \exists \alpha = \sup A = \max A \in A$.

//

Review

$$\boxed{\begin{array}{l} A \subset \mathbb{R} \neq \phi,\ \text{bdd above} \\ \Rightarrow \exists \alpha = \sup A \in \mathbb{R} \end{array}}$$

上限公理

Review

$$\boxed{\begin{array}{l} (X, \leq)\ \text{ordered set} \\ A \subset X \\ \alpha = \sup A \in A \\ \Rightarrow \alpha = \max A \end{array}}$$

Th

$A \subset \mathbb{R}$ $\neq \phi$, bdd, closed

$\Rightarrow \exists \alpha = \max A \quad (\in A)$

$\exists \beta = \min A \quad (\in A)$

Th

(X, d) MS

$\mathbb{F}$ the set that collects all closed sets of X

$\Rightarrow$ (F1) $X, \phi \in \mathbb{F}$

(F2) $F_\mu \in \mathbb{F}$ $(\mu \in M)$

$\Rightarrow \bigcap_{\mu \in M} F_\mu \in \mathbb{F}$

(F3) $F_i \in \mathbb{F}$ $(i = 1, \cdots, m)$

$\Rightarrow \bigcup_{i=1}^{m} F_i \in \mathbb{F}$

Proof

(F1) $\underline{X \in \mathbb{F}}$

i.e. $\{x_n\} \subset X : x_n \to x \in X \Rightarrow x \in X$

OK

$\underline{\phi \in \mathbb{F}}$

i.e. $\{x_n\} \subset \phi : x_n \to x \in X \Rightarrow x \in \phi$

The desired result holds since $\nexists \{x_n\} \subset \phi$.

(F2) Assume that $F_\mu \in \mathbb{F}$ $(\mu \in M)$.

We prove that $\underline{\bigcap_{\mu \in M} F_\mu \in \mathbb{F}}$.

Let $\{x_n\} \subset \bigcap_{\mu \in M} F_\mu$: $x_n \to x \in X$.

i.e. $\forall n \in \mathbb{N}$, $x_n \in \bigcap_{\mu \in M} F_\mu$

i.e. $\forall n \in \mathbb{N}$, $\mu \in M$, $x_n \in F_\mu$

Our goal is to show that $\underline{x \in \bigcap_{\mu \in M} F_\mu}$.

Let $\mu \in M$ and fix it.

As $\{x_n\} \subset F_\mu$, $x_n \to x \in X$, and

F_μ is closed in X,

we obtain $x \in F_\mu$ $(\mu \in M)$.

$\therefore x \in \bigcap_{\mu \in M} F_\mu$.

$\lrcorner$

(F3) Assume that $F_1, \cdots, F_m \in \mathbb{F}$.

We demonstrate that $\bigcup_{i=1}^{m} F_i \in \mathbb{F}$.

Let $\{x_n\} \subset \bigcup_{i=1}^{m} F_i$ such that $x_n \to x \in X$.

Our aim is to prove that $x \in \bigcup_{i=1}^{m} F_i$.

i.e. $x \in F_i$ for some $i = 1, \cdots, m$

As $\{x_n\} = \{x_1, x_2, x_3, \cdots\} \subset \bigcup_{i=1}^{m} F_i$,

$\exists\, i_0 = 1, \cdots, m$:

F_{i_0} contains infinite numbers of x_n.

Therefore, $\exists \{x_{n_k}\} \subset \{x_n\} : \{x_{n_k}\} \subset F_{i_0}$.

As $x_n \to x$, we have $x_{n_k} \to x$ $(k \to \infty)$.

As $\{x_{n_k}\} \subset F_{i_0}$, $x_{n_k} \to x \in X$, and

F_{i_0} is closed in X, we obtain $x \in F_{i_0}$.

This means that $x \in \bigcup_{i=1}^{m} F_i$.

$\therefore \bigcup_{i=1}^{m} F_i \in \mathbb{F}$.

//

ex

$\mathbb{R}$

$F_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$ $(n \in \mathbb{N})$: closed in $\mathbb{R}$

$\Rightarrow \bigcap_{n \in \mathbb{N}} F_n = \{0\}$: closed in $\mathbb{R}$.

ex

$\mathbb{R}$

$F_n' = \left[\frac{1}{n}, 2\right]$ $(n \in \mathbb{N})$: closed in $\mathbb{R}$.

$\Rightarrow \bigcup_{n \in \mathbb{N}} F_n' = (0, 2]$: not closed in $\mathbb{R}$.

ex

$(0, \infty)$

$F_n'' = \left(0, \frac{1}{n}\right]$ $(n \in \mathbb{N})$: closed in $(0, \infty)$.

$\Rightarrow \bigcap_{n \in \mathbb{N}} F_n'' = \emptyset$: closed in $(0, \infty)$.

Th

(X, d) MS

$\mathbb{G}$ the set that collects all open sets in (X, d)

$\Rightarrow$ (G1) $X, \phi \in \mathbb{G}$

(G2) $G_\mu \in \mathbb{G}\ (\mu \in M) \Rightarrow \bigcup_{\mu \in M} G_\mu \in \mathbb{G}$

(G3) $G_i \in \mathbb{G}\ (i = 1, \cdots, m) \Rightarrow \bigcap_{i=1}^{m} G_i \in \mathbb{G}$

Proof

(G1) OK.

(G2) Assume that $G_\mu \in \mathbb{G}\ (\mu \in M)$.

i.e. $G_\mu^c \in \mathbb{F}$

Then, our aim is to prove that

$\underline{\bigcup_{\mu \in M} G_\mu \in \mathbb{G}}$. i.e. $\left(\bigcup_{\mu \in M} G_\mu\right)^c \in \mathbb{F}$

It follows that

$$\left(\bigcup_{\mu \in M} G_\mu\right)^c = \bigcap_{\mu \in M} G_\mu^c \in \mathbb{F}.$$

De Morgan's law

(G3) Assume that $G_i \in \mathbb{G}$ $(i=1,\cdots,m)$.

i.e. $G_i^c \in \mathbb{F}$

We show that $\bigcap_{i=1}^{m} G_i \in \mathbb{G}$.

ie $\left(\bigcap_{i=1}^{m} G_i\right)^c \in \mathbb{F}$

It holds that

$$\left(\bigcap_{i=1}^{m} G_i\right)^c = \bigcup_{i=1}^{m} G_i^c \in \mathbb{F}.$$

//

Th

(X, d) MS

$\Rightarrow$ (F1) $X, \emptyset \in \mathbb{F}$

(F2) $F_\mu \in \mathbb{F}$ $(\mu \in M) \Rightarrow \bigcap_{\mu \in M} F_\mu \in \mathbb{F}$

(F3) $F_i \in \mathbb{F}$ $(i=1,\cdots,m) \Rightarrow \bigcup_{i=1}^{m} F_i \in \mathbb{F}$

Th

(X, d) MS

$\Rightarrow$ (G1) $X, \phi \in \mathfrak{G}$

(G2) $G_\mu \in \mathfrak{G}\ (\mu \in M) \Rightarrow \bigcup_\mu G_\mu \in \mathfrak{G}$

(G3) $G_i \in \mathfrak{G}\ (i=1, \cdots, m) \Rightarrow \bigcap_{i=1}^{m} G_i \in \mathfrak{G}$

Proof (Alternative)

(G1) OK

(G2) Assume that $G_\mu \in \mathfrak{G}\ (\mu \in M)$.

We prove that $\bigcup_\mu G_\mu \in \mathfrak{G}$.

Let $x \in \bigcup_\mu G_\mu$.

Then, $\exists \mu_0 \in M : x \in G_{\mu_0}$.

As G_{μ_0} is open in X, for $x \in G_{\mu_0}$,

$\exists r > 0 : S_r(x) \subset G_{\mu_0} \subset \bigcup_\mu G_\mu$.

$\therefore\ \forall x \in \bigcup_\mu G_\mu, \exists r > 0 : S_r(x) \subset \bigcup_\mu G_\mu$.

$\therefore\ \bigcup_\mu G_\mu \in \mathfrak{G}$. $\lrcorner$

(G3) Assume that $G_i \in \mathcal{G}$ $(i=1,\cdots,m)$.

We prove that $\bigcap_{i=1}^{m} G_i \in \mathcal{G}$.

Let $x \in \bigcap_{i=1}^{m} G_i$.

Then, $\forall i=1,\cdots,m$, $x \in G_i$.

As $x \in G_1$, $\exists r_1 > 0 : S_{r_1}(x) \subset G_1$

As $x \in G_m$, $\exists r_m > 0 : S_{r_m}(x) \subset G_m$.

Let $r = \min\{r_1, r_2, \cdots, r_m\} > 0$.

As $r \leq r_1$, $S_r(x) \subset S_{r_1}(x) \subset G_1$;

As $r \leq r_m$, $S_r(x) \subset S_{r_m}(x) \subset G_m$.

Therefore, $S_r(x) \subset \bigcap_{i=1}^{m} G_i$.

$\therefore \forall x \in \bigcap_{i=1}^{m} G_i$, $\exists r > 0 : S_r(x) \subset \bigcap_{i=1}^{m} G_i$.

//

ex

$\mathbb{R}$

$G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$: open in $\mathbb{R}$

$(n \in \mathbb{N})$

$\Rightarrow \bigcap_{n \in \mathbb{N}} G_n = \{0\}$: not open in $\mathbb{R}$.

Open and closed sets (2)

1. Xを距離空間，$A \subset C \subset X$とする．AがXの閉集合のとき，AはCにおいても閉集合であることを示せ．また，逆はいえないことを例を挙げて説明せよ．

2. 問題1の設定にCがXにおいて閉集合であるという仮定を付け加えると，逆もいえる．このことを示せ．

3. (復習) Aを$\mathbb{R}$の上に有界な部分集合で，$\alpha = \sup A$とする．このとき，

$$\forall \varepsilon > 0,\ \exists x \in A : \alpha - \varepsilon < x$$

がいえることを証明せよ．ここで，Aが上に有界という仮定はなぜ必要になっているか考えよ．

4. Aを$\mathbb{R}$の上に有界な部分集合で，$\alpha = \sup A$とする．このとき，Aの点列でαに収束してくるものが存在する．これを示せ．

5. (supとmaxの定義を再確認したうえで)次を示せ：
Aを$\mathbb{R}$の非空で上に有界な閉集合とする．このとき，$\max A$が存在する．

6. Xを距離空間，$\mathbb{F}$をXの閉部分集合をすべて集めた集合族とする．このとき，以下を示せ．

$$(F1)\quad X, \emptyset \in \mathbb{F};$$
$$(F2)\quad F_\mu \in \mathbb{F}\ (\mu \in M) \Rightarrow \bigcap_{\mu \in M} F_\mu \in \mathbb{F};$$
$$(F3)\quad F_i \in \mathbb{F}\ (i = 1, \cdots, m) \Rightarrow \bigcup_{i=1}^{m} F_\mu \in \mathbb{F}.$$

7. 問題4の$(F3)$について，閉集合の無限個の合併が閉集合にならない例を挙げよ．

8. Xを距離空間，$\mathbb{G}$をXの開部分集合をすべて集めた集合族とする．このとき，以下が成り立つ．

$$(G1)\quad X, \emptyset \in \mathbb{G};$$
$$(G2)\quad G_\mu \in \mathbb{G}\ (\mu \in M) \Rightarrow \bigcup_{\mu \in M} G_\mu \in \mathbb{G};$$
$$(G3)\quad G_i \in \mathbb{G}\ (i = 1, \cdots, m) \Rightarrow \bigcap_{i=1}^{m} G_\mu \in \mathbb{G}.$$

このことを，ド・モルガンの法則を用いて問題4の結果に帰着させるやり方と，そうではなく開集合の定義に基づくやり方の2通りの方法で証明せよ．

9. 問題6の$(G3)$について，開集合の無限個の共通部分が開集合にならない例を挙げよ．