Convergence of sequences

Review

(X, d) M.S

(d1) $d(x,y) \geqq 0$; $d(x,y)=0 \Leftrightarrow x=y$

(d2) $d(x,y)=d(y,x)$

(d3) $d(x,y) \leqq d(x,z)+d(z,y)$

- $S_\varepsilon(x)=\{y \in X \mid d(x,y)<\varepsilon\}$ $(\subset X)$.

  open sphere

- $|d(x,y)-d(x,z)| \leqq d(y,z)$.

$\{x_n\} \subset X$ <u>sequence</u> in X $(\neq \emptyset)$

$\iff f : \mathbb{N} \to X$

Def

(X, d) MS

$x \in X$

$\{x_n\} \subset X$

$\{x_n\}$ converges to x

$\iff x_n \to x$

$\iff d(x_n, x) \to 0$

$\iff \forall_{\varepsilon > 0}, \exists_{n_0 \in \mathbb{N}} : n \geq n_0 \Rightarrow d(x_n, x) < \varepsilon$

negation

$x_n \not\to x$

$\iff \exists_{\varepsilon > 0} : \forall_{n \in \mathbb{N}},$

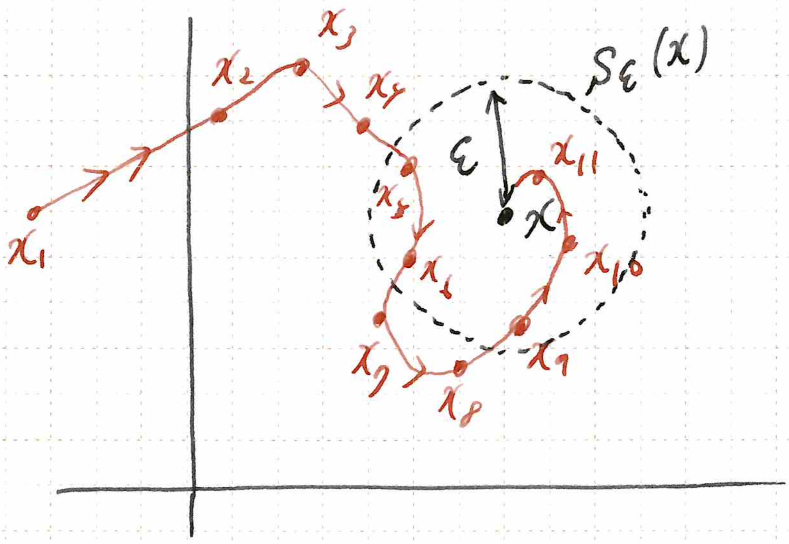
$\exists_{n' \geq n} : d(x_{n'}, x) \geq \varepsilon.$

$x_n \longrightarrow x$

$\Longleftrightarrow \forall_{\varepsilon>0},\ \exists n_0 \in \mathbb{N}:$

$$n \geq n_0 \Longrightarrow d(x_n, x) < \varepsilon$$

$$\Longleftrightarrow x_n \in S_\varepsilon(x)$$



ex

$\mathbb{R}^2$

$$d((x,y),(u,v)) = \sqrt{(x-u)^2+(y-v)^2}$$

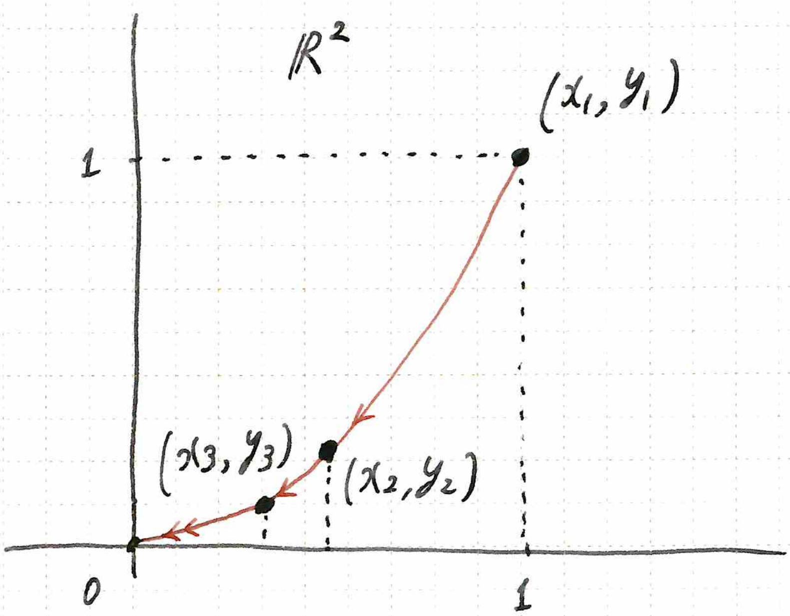
Let $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^2}\right)$ $(n \in \mathbb{N})$.

i.e. $(x_1, y_1) = (1, 1)$

$(x_2, y_2) = \left(\frac{1}{2}, \frac{1}{4}\right)$

$\vdots$

Then, $(x_n, y_n) \to (0, 0)$.



(X, d) MS

$\{x_n\} \subset X$, $x, y \in X$

$x_n \to x$

$x_n \to y$

$\Rightarrow x = y$

<u>Proof.</u>

It holds that

$$0 \leq d(x, y)$$
$$\leq d(x, x_n) + d(x_n, y) \to 0.$$

Thus, $d(x, y) = 0$. $\therefore x = y$. //

<u>Review</u>

$\{a_n\}, \{b_n\} \subset \mathbb{R}$

$a_n \to a$

$b_n \to b$

$\Rightarrow a_n + b_n \to a + b$

$\{a_n\}, \{b_n\}, \{c_n\} \subset \mathbb{R}$

$a_n \leq b_n \leq c_n$

$a_n \to a$

$c_n \to a$

$\Rightarrow b_n \to a$

※ $x_n \to x \Leftrightarrow x = \lim x_n$

Def

$$\boxed{\begin{array}{l}\{x_n\} \subset X \ \underline{\text{convergent}} \\ \Longleftrightarrow \exists x \in X : x_n \to x\end{array}}$$

$$\Longleftrightarrow \exists x \in X ; \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N} :$$
$$n \geqq n_0 \Rightarrow d(x_n, x) < \varepsilon$$

$\Big\downarrow$ negation

$\{x_n\} \subset X$ is <u>not</u> convergent.

$$\Longleftrightarrow \forall x \in X, \ \exists \varepsilon > 0 : \forall n \in \mathbb{N}, \ \exists n' \geqq n :$$
$$d(x_{n'}, x) \geqq \varepsilon .$$

ex

$\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\} \subset \mathbb{R}$

- $\{x_n\}$ is convergent in $\mathbb{R}$.

  $(\because)$ $\exists_{0 \in \mathbb{R}}: x_n \left(= \frac{1}{n}\right) \to 0.$

- $\{x_n\} \subset (0, 1]$ is not convergent.

  $(\because)$ $\forall_{x \in (0,1]},$

  $\exists \varepsilon > 0 : \forall_{n \in \mathbb{N}}, \exists_{n' \geqq n} :$

  $\left| x - \frac{1}{n'} \right| \geqq \varepsilon .$

ex

$\{x_n\} = \{3, 3.1, 3.14, 3.141, \cdots\} \subset \mathbb{Q} \subset \mathbb{R}$

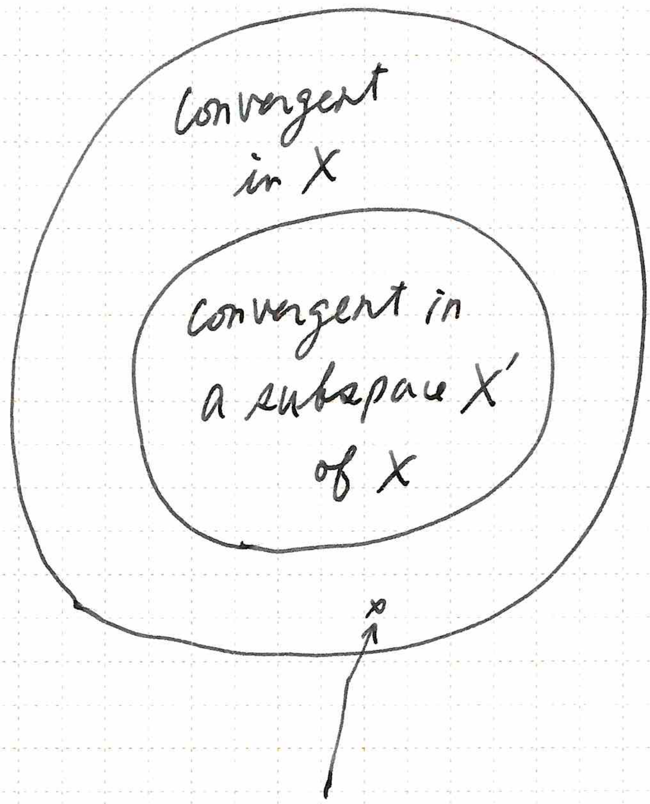
- $\{x_n\}$ is convergent in $\mathbb{R}$.
- $\{x_n\}$ is not convergent in $\mathbb{Q}$.

X MS

$X' \subset X$ subspace of X

$\{x_n\} \subset X'$ convergent

$\Rightarrow \{x_n\}$: convergent in X.



$$\{x_n\} = \left\{\frac{1}{n}\right\} \subset \underset{\underset{X'}{\parallel}}{(0,1]} \subset \underset{\underset{X}{\parallel}}{\mathbb{R}}$$

(X, d) metric space

$\{x_n\}, \{y_n\} \subset X$

$x, y \in X$

$x_n \to x$ i.e. $d(x_n, x) \to 0$

$y_n \to y$ i.e. $d(y_n, y) \to 0$

$\Rightarrow d(x_n, y_n) \to d(x, y)$

Proof

We show that $|d(x_n, y_n) - d(x, y)| \to 0$.

It holds that

$$
\begin{aligned}
&|d(x_n, y_n) - d(x, y)| \\
&\quad \leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \\
&\quad \leq d(y_n, y) + d(x_n, x) \\
&\quad \to 0.
\end{aligned}
$$

//

※ $\lim d(x_n, y_n) = d(\lim x_n, \lim y_n)$

(X, d) metric space

$x \in X$

$\{x_n\} = \{x, x, x, \cdots\}$

$\Rightarrow x_n \to x$

Proof

It follows that

$$
\begin{aligned}
d(x_n, x) &= d(x, x) \\
&= 0 \quad \forall n \in \mathbb{N}.
\end{aligned}
$$

$\therefore d(x_n, x) \to 0$

//

(d1) $d(x, y) \geqq 0$; $d(x, y) = 0 \Leftrightarrow x = y$

(d2) $d(x, y) = d(y, x)$

(d3) $d(x, y) \leqq d(x, z) + d(z, y)$

(X, d) discrete MS

$\{x_n\} \subset X$, $x \in X$

$\Rightarrow$ Equivalent

① $x_n \to x$

② $\exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow x_n = x$

Proof.

①$\Rightarrow$②

From ①, for $\varepsilon = 1/2 > 0$,

$\exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow d(x_n, x) < \frac{1}{2}$. —(*)

As d is the discrete metric,

(*) means that $x_n = x$ $(n \geqq n_0)$.

Thus, ② follows. ⌟

②$\Rightarrow$①

OK.

//

ex

$\{x_n\} = \{\frac{1}{n}\} \subset \mathbb{R}$

Then, $x_n \to 0$

ex

$\mathbb{R}$ with the discrete metric

$\{x_n\} = \{\frac{1}{n}\}$

Then, $x_n \not\to 0$

ex

$X = \{$グー, チョキ, パー$\}$

with the discrete metric

$\{x_n\} = \{$グー, グー, グー, $\cdots\}$

Then, $x_n \to$ グー.

Def

$X \neq \phi$

$\{a_n\} \subset X$

$\{a_{n_i}\} = \{a_{n_1}, a_{n_2}, a_{n_3}, \cdots\}$

<u>subsequence</u> of $\{a_n\}$

$\Leftrightarrow$ ① $\{a_{n_i}\} \subset \{a_n\}$

② $n_i \in \mathbb{N}$ with $n_1 < n_2 < n_3 < \cdots$

ex

$\{\underline{a_1}, \underline{a_2}, a_3, \underline{a_4}, a_5, a_6, \underline{a_7}, \underline{a_8}, a_9, \cdots\}$

$$
\begin{array}{ccccccccc}
\bullet\ \{a_1, & a_2, & a_4, & a_7, & a_8, & a_{11}, & a_{14}, & \cdots\} \\
\parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \\
a_{n_1} & a_{n_2} & a_{n_3} & a_{n_4} & a_{n_5} & a_{n_6} & a_{n_7} &
\end{array}
$$

a subsequence of $\{a_n\}$

• $\{a_1, a_1, a_2, a_4, a_7, a_7, a_7, a_8, \cdots\}$

これはダメ!

$n_1 < n_2 < n_3 < \cdots$ をみたさない。

※ $n_i \to \infty$ (as $i \to \infty$)

※ $\{a_n\}$ is a subsequence of $\{a_n\}$.

ex

$X = \mathbb{R}$

$\{a_n\} = \{1, 0, 1, 0, 1, 0, \cdots\}$

Then, $\{a_{n_i}\} = \{0, 0, 0, \cdots\}$

is a subsequence of $\{a_n\}$.

ex

$X = \mathbb{R}$

$\{a_n\} = \{1, 0, 2, 0, 3, 0, \cdots\}$

Then, $\{a_{n_i}\} = \{1, 2, 3, \cdots\}$

is a subsequence of $\{a_n\}$.

Let $\{a_{n_{i_j}}\} = \{2, 4, 6, 8, \cdots\}$.

Then, • $\{a_{n_{i_j}}\}$ is a subsequence of $\{a_{n_i}\}$.

• $\{a_{n_{i_j}}\}$ is a subsequence of $\{a_n\}$.

$\left(\{a_{n_{i_j}}\} \subset \{a_{n_i}\} \subset \{a_n\}. \right)$.

X MS
$\{a_n\} \subset X$
$a \in X$
$\Rightarrow$ Equivalent
① $a_n \to a$
② $^\forall \{a_{n_i}\} \subset \{a_n\}$, $a_{n_i} \to a$

Proof

① $\Rightarrow$ ②

Let $\{a_{n_i}\}$ be a subsequence of $\{a_n\}$.

We prove that $a_{n_i} \to a$.

i.e. $^\forall \varepsilon > 0, ^\exists i_0 \in \mathbb{N} : i \geq i_0 \Rightarrow d(a_{n_i}, a) < \varepsilon$

Let $\varepsilon > 0$.

From ①, $^\exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow d(a_n, a) < \varepsilon$. —(*)

For $n_0 \in \mathbb{N}$, $^\exists i_0 \in \mathbb{N} : i \geq i_0 \Rightarrow n_0 \leq n_i$. —(**)

Let $i \geq i_0$.

From (*) and (**), $d(a_{n_i}, a) < \varepsilon$.

$\therefore {}^\forall \varepsilon > 0, ^\exists i_0 \in \mathbb{N} : i \geq i_0 \Rightarrow d(a_{n_i}, a) < \varepsilon$.

$\therefore a_{n_i} \to a$.

② $\Rightarrow$ ①

OK.

//

X MS
$\{a_n\} \subset X$, $a \in X$
$\Rightarrow$ Equivalent

① $a_n \not\to a$

i.e. $\exists \varepsilon > 0 : \forall n \in \mathbb{N}, \exists n' \geqq n : d(a_{n'}, a) \geqq \varepsilon$

② $\exists \varepsilon > 0, \{a_{n_i}\} \subset \{a_n\}$:

$\forall i \in \mathbb{N}, d(a_{n_i}, a) \geqq \varepsilon$

Proof

①⇒②

From ①, $\exists \varepsilon > 0$:

for $n = 1$, $\exists n_1 \geqq 1 :$ $d(a_{n_1}, a) \geqq \varepsilon$;

for $n = n_1 + 1$, $\exists n_2 \geqq n_1 + 1 > n_1 : d(a_{n_2}, a) \geqq \varepsilon$;

for $n = n_2 + 1$, $\exists n_3 \geqq n_2 + 1 > n_2 : d(a_{n_3}, a) \geqq \varepsilon$;

.....

We obtain $\{a_{n_i}\} \subset \{a_n\}$:

$\forall i \in \mathbb{N}, d(a_{n_i}, a) \geqq \varepsilon$.

②⇒①

From ②, $\exists \{a_{n_i}\} \subset \{a_n\} : a_{n_i} \not\to a$.

$\therefore a_n \not\to a$.

//

X MS

$\{a_n\} \subset X$, $a \in X$

$\Rightarrow$ Equivalent

① $a_n \to a$

② $\forall \{a_{n_i}\} \subset \{a_n\}$, $a_{n_i} \to a$

③ $\forall \{a_{n_i}\} \subset \{a_n\}$, $\exists \{a_{n_j}\} \subset \{a_{n_i}\} : a_{n_j} \to a$

Proof

①⇒② OK

②⇒③ Obvious

③⇒①

Suppose by contradiction that $a_n \not\to a$.

Then, $\exists \varepsilon > 0$, $\{a_{n_i}\} \subset \{a_n\}$:

$$\forall i \in \mathbb{N},\ d(a_{n_i}, a) \geq \varepsilon. \quad — (*)$$

Let $\{a_{n_j}\} \subset \{a_{n_i}\}$.

From (*), $a_{n_j} \not\to a$.

$\therefore \exists \{a_{n_i}\} \subset \{a_n\}$:

$$\forall \{a_{n_j}\} \subset \{a_{n_i}\},\ a_{n_j} \not\to a.$$

This contradicts ③.

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Convergence of sequences

1. 距離空間内の点列が収束することとしないことの定義を述べ，例を挙げながら説明せよ．

2. $\mathbb{R}^2$において，

$$(x_n, y_n) = \left(\frac{1}{n}\cos\frac{n}{2}\pi,\ \frac{1}{n}\sin\frac{n}{2}\pi\right) \quad (n \in \mathbb{N})$$

という点列を考える．(x,y)−平面上に$(x_1,y_1),\cdots,(x_4,y_4)$をプロットし，この点列が原点に収束することを直観的に納得せよ．

3. 距離空間において，極限が一意に定まることを証明せよ．また，その際に，実数空間における数列の収束に関する結果を用いたなら，それを（証明も含めて）復習せよ．

※以後，特に断らなくても，以前証明した結果を使うときはコマメに復習して完全マスターせよ．

4. Xを距離空間，Yをその部分空間とする．Yにおける点列$\{x_n\}$が（Yにおいて）収束するならば，$\{x_n\}$はXにおいても収束する．このことを示せ．

5. Xを距離空間，$\{x_n\}, \{y_n\}$をXの点列，x, yをXの要素とする．このとき，

$$x_n \to x,\ y_n \to y \Rightarrow d(x_n, y_n) \to d(x,y)$$

がいえる．これを示せ．

6. 離散距離空間においては，点列が収束することは，ある番号から先はその点列の項がその極限に一致することと同値である．このことを証明せよ．

7. Xを距離空間，$\{x_n\}$をXの点列，xをXの要素とする．このとき，$\{x_n\}$がxに収束しないことは，次の条件と同値である：

$$\text{ある}\varepsilon > 0\text{と}\{x_n\}\text{の部分列}\{x_{n_i}\}\text{が存在し，}$$
$$\text{任意の番号}i \in \mathbb{N}\text{について}d(x_{n_i}, x) \geq \varepsilon\text{となる．}$$

このことを証明せよ．

8. Xを距離空間，$\{x_n\}$をXの点列，xをXの要素とする．このとき，以下の3条件が同値であることを示せ．

(i) $x_n \to x$;

(ii) $\forall \{x_{n_i}\} \subset \{x_n\},\ x_{n_i} \to x$;

(iii) $\forall \{x_{n_i}\} \subset \{x_n\},\ \exists \{x_{n_j}\} \subset \{x_{n_i}\} : x_{n_j} \to x$.