Bounded functions

Review

$(X, \leq)$ ordered set

$A \subset X$, $\neq \emptyset$

- A: <u>bounded from above</u>

  (bdd above)

  $\Leftrightarrow \exists \underset{\sim}{M} \in X : \forall_{x \in A}, x \leq M$

  an upper bound of A

- A: <u>bounded</u> (bdd)

  $\Leftrightarrow$ A is bdd above and bdd below

Def

$X \neq \phi$

$f : X \to \mathbb{R}$

- f is <u>bounded from above</u>.
  (bdd above)

$\Leftrightarrow$ $f(X)\ (\subset \mathbb{R})$: bdd above

$\Leftrightarrow$ $\exists M \in \mathbb{R} : \forall x \in X,\ f(x) \leqq M$

$\Leftrightarrow$ $\exists M \geqq 0 : \forall x \in X,\ f(x) \leqq M$

$\Leftrightarrow$ $\exists M > 0 : \forall x \in X,\ f(x) \leqq M$

Def

- f is <u>bounded</u> (bdd).

$\Leftrightarrow$ f is bdd above and bdd below.

$\Leftrightarrow$ $\exists L, M \in \mathbb{R} ; \forall x \in X,\ L \leqq f(x) \leqq M$

$\Leftrightarrow$ $\exists M \in \mathbb{R} : \forall x \in X,\ |f(x)| \leqq M$

- $f: X \to \mathbb{R}$ bdd above

$\iff {}^\exists M \geqq 0 : {}^\forall x \in X,\ f(x) \leqq M$

$\Big\downarrow$ negation (否定)

$f: X \to \mathbb{R}$ is not bdd above.

$\iff {}^\forall M \geqq 0,\ {}^\exists x \in X : \ M < f(x)$

- $f: X \to \mathbb{R}$ bdd

$\iff f$: bdd above and bdd below

$\Big\downarrow$ negation

$f: X \to \mathbb{R}$ is not bdd.

$\iff f$ is not bdd above $\boxed{\text{or}}$

f is not bdd below

ex

$X = (0, \infty)$ $(\subset \mathbb{R})$

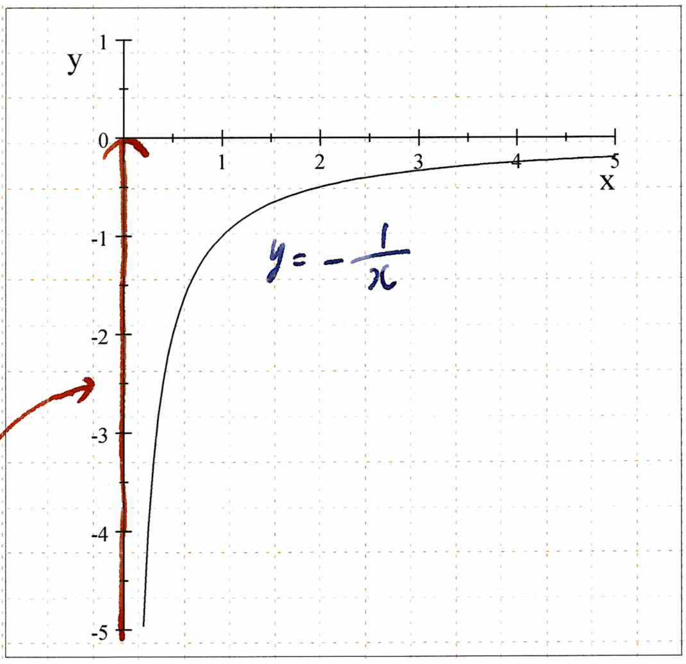
$f : X \to \mathbb{R}$ defined as follows :

$$f(x) = \frac{-1}{x} \quad \forall x \in X.$$

Then, • f is bdd above.

• f is not bdd below

• f is not bdd.



$f(X) = (-\infty, 0)$

Def

- $\{a_n\} \subset \mathbb{R}$ <u>bdd above</u>.

  $\Leftrightarrow \exists M \geq 0 : \forall n \in \mathbb{N},\ a_n \leq M$

- $\{a_n\} \subset \mathbb{R}$ <u>bdd</u>

  $\Leftrightarrow \{a_n\}$ is bdd above and bdd below.

  $\Leftrightarrow \exists M \geq 0 : \forall n \in \mathbb{N},\ |a_n| \leq M$

ex

- $\{a_n\} = \{1, 2, 3, 4, \cdots\}$ $(\subset \mathbb{R})$.

  Then, • $\{a_n\}$ is not bdd above.

  • $\{a_n\}$ is bdd below.

  • $\{a_n\}$ is not bdd.

- $\{b_n\} = \{1, 0, 1, 0, \cdots\}$

  Then, $\{b_n\}$ is bdd

- $\{c_n\} = \{1, -1, 2, -2, 3, -3, \cdots\}$

  Then, • $\{c_n\}$ is not bdd above.

  • $\{c_n\}$ is not bdd below.

  • $\{c_n\}$ is not bdd.

$X \neq \emptyset$

$f, g : X \to \mathbb{R}$ bdd

$\Rightarrow f+g$: bdd

Proof

As f is bdd, ${}^{\exists}M_1 \geqq 0 : {}^{\forall}x \in X, |f(x)| \leqq M_1$.

As g is bdd, ${}^{\exists}M_2 \geqq 0 : {}^{\forall}x \in X, |g(x)| \leqq M_2$.

Define $M \equiv M_1 + M_2 \geqq 0$.

Let $x \in X$.

Then, it holds true that

$$
\begin{aligned}
&|(f+g)(x)| \\
&\quad = |f(x) + g(x)| \\
&\quad \leqq |f(x)| + |g(x)| \\
&\quad \leqq M_1 + M_2 \equiv M.
\end{aligned}
$$

$\therefore {}^{\exists}M \geqq 0 : {}^{\forall}x \in X, |(f+g)(x)| \leqq M$.

This shows that $f+g$ is bdd.

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$X \neq \phi$

$f, g : X \to \mathbb{R}$ bdd above ← weak

$\Rightarrow f+g$: bdd above ← weak

$X \neq \emptyset$

$f : X \to \mathbb{R}$ bdd

$\alpha \in \mathbb{R}$

$\Rightarrow \alpha f$: bdd

<u>Proof</u>

As f is bdd,

$\exists L \geqq 0 : \forall x \in X, \ |f(x)| \leqq L.$

Define $M \equiv |\alpha| L \geqq 0$.

Let $x \in X$.

It holds that

$$
\begin{aligned}
&|(\alpha f)(x)| \\
&\quad = |\alpha \cdot f(x)| \\
&\quad = |\alpha| \cdot |f(x)| \\
&\quad \leqq |\alpha| \cdot L \equiv M.
\end{aligned}
$$

$\therefore \exists M \geqq 0 : \forall x \in X, \ |\alpha f(x)| \leqq M.$

//

$$X \neq \emptyset$$
$$f : X \to \mathbb{R} \text{ bdd above}$$ weak
$$\alpha \geqq 0$$
$$\Rightarrow \alpha f : \text{bdd above}$$ weak

Proof

We prove that

$$\exists M \geqq 0 : \forall x \in X, \alpha f(x) \leqq M.$$

As f is bdd above,

$$\exists L \geqq 0 : \forall x \in X, f(x) \leqq L.$$

As $\alpha \geqq 0$, we have $\alpha f(x) \leqq \alpha L = M$.

Therefore,

$$\exists M \geqq 0 : \forall x \in X, \alpha f(x) \leqq M.$$

//

Th

$X \neq \phi$

$f, g : X \to \mathbb{R}$ bdd

$\alpha, \beta \in \mathbb{R}$

$\Rightarrow \alpha f + \beta g$: bdd

線型性

Proof

As f is bdd, αf is also bdd.

Similarly, as g is bdd, so is βg.

Consequently, $\alpha f + \beta g$ is bdd.

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Cor

$X \neq \emptyset$

$f, g, h : X \to \mathbb{R}$ bdd

$\alpha, \beta, \gamma \in \mathbb{R}$

$\Rightarrow \alpha f + \beta g + \gamma h$: bdd

Cor

$X \neq \emptyset$

$f, g : X \to \mathbb{R}$ bdd

$\Rightarrow f - g$: bdd

Review

$(X, \leqq)$ ordered set

$A \subset X, \neq \phi$

$\alpha = \sup A \in X$

$\Leftrightarrow \begin{cases} ① \ \forall x \in A, \ x \leqq \alpha \\ ② \ \forall x \in A, \ x \leqq a \Rightarrow \alpha \leqq a \end{cases}$

Axiom

$A \subset \mathbb{R} \neq \phi$, bdd above

$\Rightarrow \exists \sup A \in \mathbb{R}$

ex

$(\mathbb{R}, \leq)$

$A = \{3, 3.1, 3.14, 3.141, \cdots\} \subset \mathbb{R}$

A : bdd above

Then, $\sup A = \pi \in \mathbb{R}$.

ex

$(\mathbb{Q}, \leq)$

$A = \{3, 3.1, 3.14, \cdots\} \subset \mathbb{Q}$

A : bdd above

Then, $\nexists \sup A \in \mathbb{Q}$

Def.

$X \neq \phi$

$f: X \to \mathbb{R}$ bdd above

- $\alpha = \sup_{x \in X} f(x)$

$= \sup f(X) \in \mathbb{R}$

$$\Longleftrightarrow \begin{cases} ① \ \forall x \in X,\ f(x) \leqq \alpha \\ ② \ \forall x \in X,\ f(x) \leqq a \Rightarrow \alpha \leqq a \end{cases}$$

※ f is not bdd above

$\Rightarrow \sup f(x) = \infty$

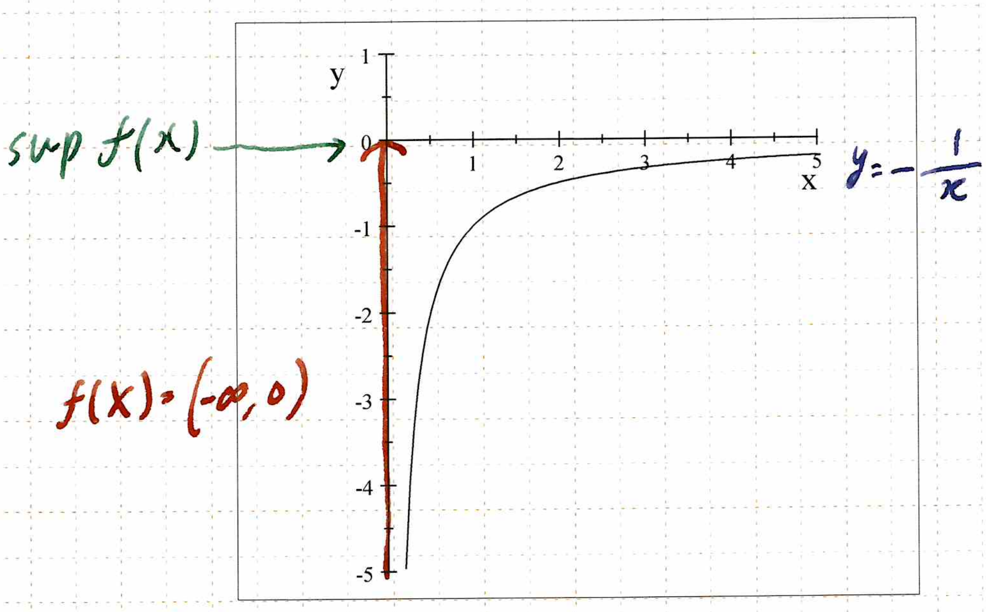
ex

$X = (0, \infty) \subset \mathbb{R}$

$f : X \to \mathbb{R}$ defined by

$$f(x) = \frac{-1}{x} \quad \forall x \in X$$

Then,
- $\sup f(x) = 0$
- $\inf f(x) = -\infty$



Def.

$\{a_n\} \subset \mathbb{R}$ bdd above

- $\alpha = \sup_{n\in\mathbb{N}} a_n$

$= \sup a_n$

$= \sup\{a_n \mid n\in\mathbb{N}\} \in \mathbb{R}$

$$\Longleftrightarrow \begin{cases} ① \ \forall n\in\mathbb{N},\ a_n \leq \alpha \\ ② \ \forall n\in\mathbb{N},\ a_n \leq a \Rightarrow \alpha \leq a \end{cases}$$

※ $\{a_n\}$ is not bdd above.

$\Rightarrow \alpha = \sup a_n = \infty$

Def

$X \neq \emptyset$

$f : X \to \mathbb{R}$ bdd above

- $\alpha = \max_{x \in X} f(x)$

$\Leftrightarrow \alpha = \max f(X)$

$\Leftrightarrow \begin{cases} ① \ \alpha \in f(X) \\ ② \ a \in f(X) \Rightarrow a \leq \alpha \end{cases}$

$\Leftrightarrow \begin{cases} ① \ \exists_{x \in X} : \alpha = f(x) \\ ② \ \forall_{x \in X}, f(x) \leq \alpha \end{cases}$

Review

$(X, \leq)$ ordered set

$A \subset X, \neq \emptyset$

$\alpha = \max A$

$\Leftrightarrow \begin{cases} ① \ \alpha \in A \\ ② \ \forall_{x \in A}, x \leq \alpha \end{cases}$

ex

$X = (0, \infty)$

$f(x) = -\frac{1}{x}$

Then, • $\sup_{x \in X} f(x) = 0$

• $\inf_{x \in X} f(x) = -\infty$

• $\nexists \max_{x \in X} f(x)$

ex

$X = \mathbb{R}$

$f(x) = \sin x$

Then, • $\sup_{x \in \mathbb{R}} f(x) = \max_{x \in \mathbb{R}} f(x) = 1$

• $\inf f(x) = \min f(x) = -1$

ex $\{a_n\} = \{3, 3.1, 3.14, \cdots\} \subset \mathbb{R}$

Then, • $\sup \{a_n\} = \pi$

• $\nexists \max \{a_n\}$

• $\inf a_n = \min a_n = 3$

Bounded functions

1. 実数値関数が上に有界であることとそうではないことの定義を述べよ．また，そのような関数の例を挙げよ．

2. 数列が上に有界であることとそうではないことの定義を述べよ．また，そのような数列の例を挙げよ．

3. 実数値関数が有界であることとそうではないことの定義を述べよ．また，そのような関数の例を挙げよ．

4. 数列が有界であることとそうではないことの定義を述べよ．また，そのような数列の例を挙げよ．

5. 有界ではない数列$\{a_n\}$と$\{b_n\}$について，$\{a_n + b_n\}$は有界になるようなものの例を挙げよ．

6. Xを空ではない集合，$f, g : X \to \mathbb{R}$を有界関数，$\alpha, \beta \in \mathbb{R}$とする．$\alpha f + \beta g$も有界関数であることを証明せよ．また，上界として何を取ることができるか？

7. Xを空ではない集合，$f, g, h : X \to \mathbb{R}$を有界関数，$\alpha, \beta, \gamma \in \mathbb{R}$とする．問題6の結果を認めた上で，$\alpha f + \beta g + \gamma h$も有界関数であることを証明せよ．また，上界として何を取ることができるか？

8. Xを空ではない集合，$f, g : X \to \mathbb{R}$を有界関数とする．$f - g$も有界関数であることを証明せよ．また，上界として何を取ることができるか？

9. 次の数列について，上限，下限，最大値，最小値を求めよ（存在しない場合もある）．

$$\begin{aligned}
&(1)\ \{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\}, \\
&(2)\ \{b_n\} = \{10, 9, 8, 7, \cdots\}, \\
&(3)\ c_n = \begin{cases} 1 - \frac{1}{n} & n = 1, 3, 5, \cdots \\ -1 + \frac{1}{n} & n = 2, 4, 6, \cdots \end{cases}
\end{aligned}$$