Sets and set inclusion (1)

$A = \{\underset{\sim}{a}, b, c\}$

an **element** of A

$a, b, c \in A$

Def

A, B sets

$A \subset B$ $\quad$ A is a **subset** of B.

(B contains A.)

$\iff \forall x \in A,\ x \in B$

$\iff x \in A \Rightarrow x \in B$

ex

$\mathbb{N} = \{1, 2, 3, \cdots\}$

$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$

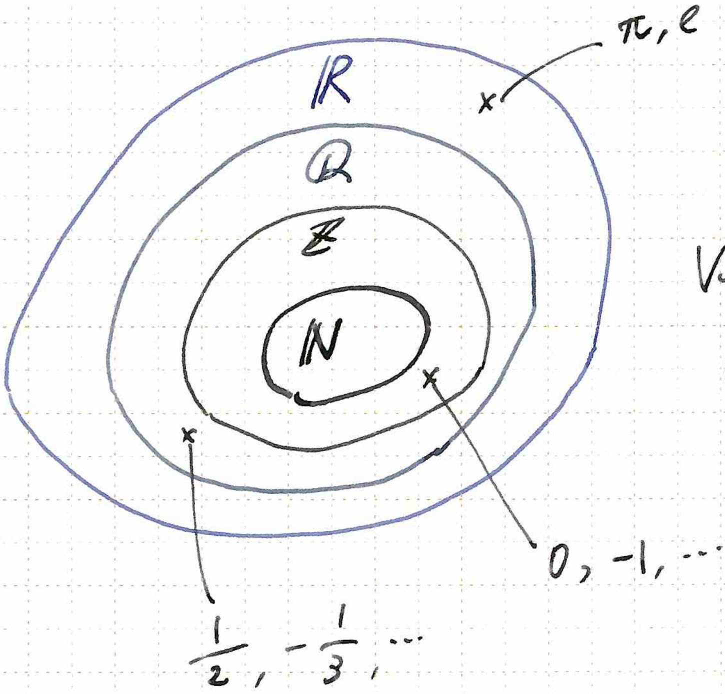
$\mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; n \in \mathbb{N},\ m \in \mathbb{Z} \right\}$

$\mathbb{R}$ the set of real numbers

Then, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

$\pi, e \in \mathbb{R}$

$\pi, e \notin \mathbb{Q}$



Venn diagram
(ベン図)

ex

$a, b \in \mathbb{R}$

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

$(a,b] = \{x \in \mathbb{R} \mid a < x \leqq b\}$

$[a,b] = \{x \in \mathbb{R} \mid a \leqq x \leqq b\}$

$(a,\infty) = \{x \in \mathbb{R} \mid a < x\}$

$(-\infty,\infty) = \mathbb{R}$

Then, $(a,b) \subset [a,b) \subset [a,b] \subset [a,\infty)$

ex

$A = \{a, \{a\}, \{a,b\}, c, d, \{e\}\}$

Then,
- $a \in A$
- $\{a\} \in A$
- $\{a\} \subset A$
- $\{\{a\}\} \subset A$
- $b \notin A$
- $\{b\} \notin A$
- $\{a,b\} \in A$
- $\{a,b\} \not\subset A$
- $\{c,d,e\} \not\subset A$
- $\{c,d,\{e\}\} \subset A$

Def

$\boxed{\phi \quad \underline{\text{empty set}}}$

ex

$$\{x \in \mathbb{R} \mid x^2 = -1\} = \phi$$

$$\boxed{\begin{array}{l} \phi \subset A \\ \text{where } A \text{ is any set.} \end{array}}$$

i.e. $x \in \phi \Rightarrow x \in A$

この前提が決して満たされない。

この場合、この条件命題

$$x \in \phi \Rightarrow x \in A$$

は真とみなす。

Def

A, B sets

$A = B$

$\Longleftrightarrow A \subset B$ and $A \supset B$

ex

$$\{a, b, c\} = \{c, b, a\}$$

※集合はものの集まりであって、
書き並べる順序は問われない。

Def

X set

2^X = the set of all subsets of X

power set of X

ex

$X = \{a, b\}$ ← 2つの要素

Then, $2^X = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ ← 4つの要素 $= 2^2$

ex

$X = \{a, b, c\}$ ← 3つ

Then, $2^X = \{\phi, \{a\}, \{b\}, \{c\},$
$\{a, b\}, \{b, c\}, \{c, a\}, X\}$ ← 8つ $= 2^3$

ex

$X = \{a\}$ ← 1つ

Then, $2^X = \{\phi, \{a\}\}$ ← 2つ $= 2^1$

ex

$X = \phi$ ← 0個

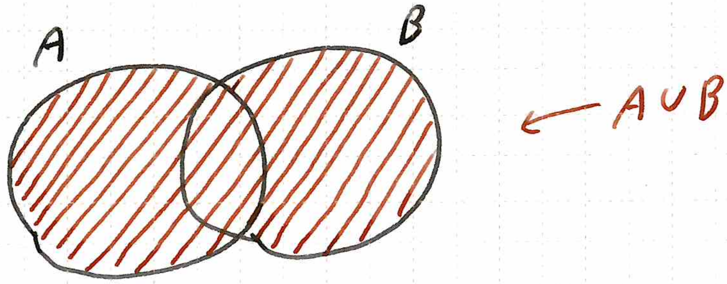
Then, $2^X = \{\phi\}$. ← 1つ $= 2^0$

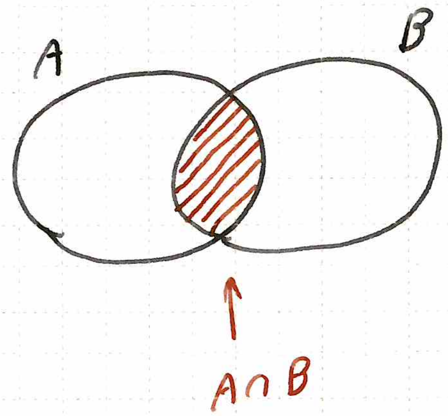
Def.

A, B sets

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$





ex

$X = \{a, b, c, d, e\}$

$A = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\} \subset 2^X$

$B = \{\emptyset, \{b\}, \{a, b\}, \{a, b, d\}, X\} \subset 2^X$

Then,

$$A \cup B = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\} \subset 2^X$$

$$A \cap B = \{\emptyset, \{a, b\}, X\} \subset 2^X$$

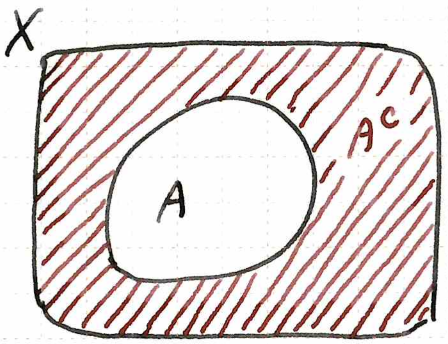
Def

X set, $\neq \phi$

$A \subset X$

$A^c = \{x \in X \mid x \notin A\}$ complement

※ X : universal set



ex

$X = \{a, b, c, d, e\}$

$A = \{a, b\} \subset X$

Then, $A^c = \{c, d, e\}$

ex

$X' = \{a, b, c\}$

$A = \{a, b\} \subset X'$

Then, $A^c = \{c\}$.

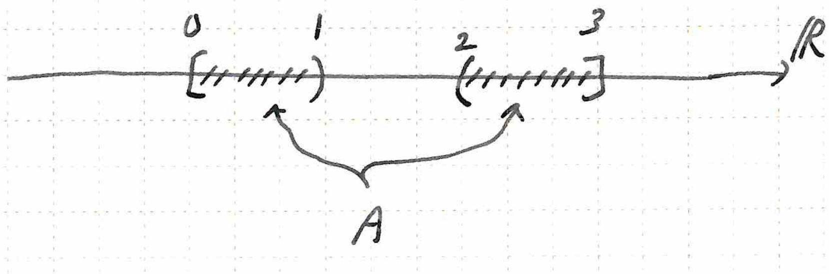
※全体集合が変われば補集合も変わってくる。

ex

$\mathbb{R}$

$A = [0, 1) \cup (2, 3]$

Then, $A^c = (-\infty, 0) \cup [1, 2] \cup (3, \infty)$



X set

$A, B \subset X$

Then, $A \subset B$

$\iff A^c \supset B^c$

Proof

$A \subset B$

$\iff x \in A \Rightarrow x \in B$

$\iff x \notin B \Rightarrow x \notin A$

$\iff x \in B^c \Rightarrow x \in A^c$

$\iff B^c \subset A^c$

//

X set

$A, B \subset X$

$\Rightarrow$ Equivalent

① $A \cap B = \phi$

② $A \subset B^c$

Proof

① $\Rightarrow$ ②

Let $x \in A$.

We show that $x \in B^c$.

i.e. $x \notin B$.

As $x \in A$, it follows from ① that $x \notin B$.

② $\Rightarrow$ ①

Suppose by contradiction that

$A \cap B \neq \phi$.

Let $x \in A \cap B$.

As $x \in A$, ② implies that $x \in B^c$.

This contradicts $x \in A \cap B$.

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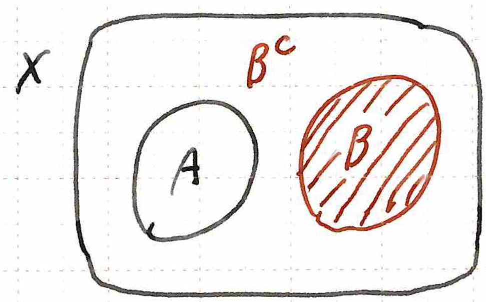
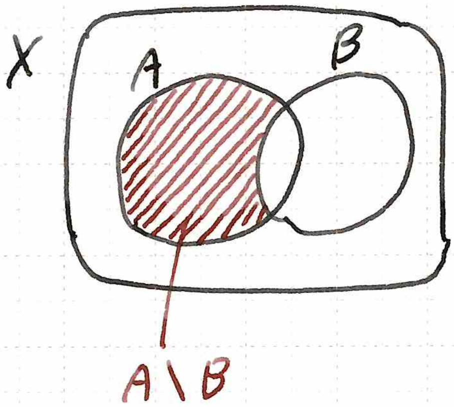


Def.

X set, $\neq \phi$

$A, B \subset X$

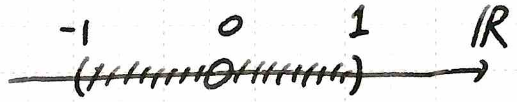
$$A \setminus B = \{x \in X \mid x \in A, x \notin B\}$$
$$= A \cap B^c$$



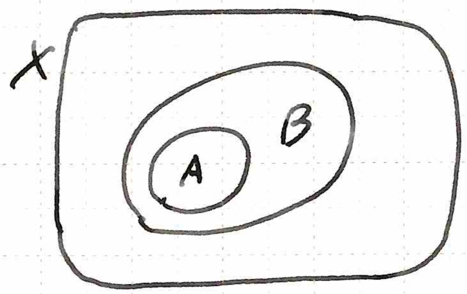
$= A - B$ と書くこともある。

ex

$$(-1,1)\setminus\{0\}$$
$$= (-1,0)\cup(0,1)$$



$$\boxed{\begin{aligned} &A\setminus B = \phi \\ &\iff A\cap B^c = \phi \\ &\iff A\subset B \end{aligned}}$$



Def.

$A_1, A_2, A_3, \cdots$ sets

- $$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$$
$$= A_1 \cup A_2 \cup A_3 \cup \cdots$$
$$= \{x \mid \exists n \in \mathbb{N} : x \in A_n\}$$

- $$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n\in\mathbb{N}} A_n$$
$$= A_1 \cap A_2 \cap A_3 \cap \cdots$$
$$= \{x \mid \forall n \in \mathbb{N},\ x \in A_n\}$$

cf. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

ex

$\mathbb{R}$

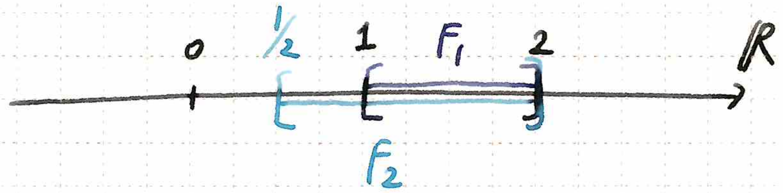
$$F_n = \left[\frac{1}{n}, 2\right] \quad (n \in \mathbb{N})$$

$$F_1 = [1, 2]$$

$$F_2 = \left[\frac{1}{2}, 2\right]$$

$$F_3 = \left[\frac{1}{3}, 2\right]$$

....



In this case,

$$\bigcap_{n=1}^{\infty} F_n = F_1 \cap F_2 \cap F_3 \cap \cdots$$

$$= [1, 2].$$

$\bigcup_{n=1}^{\infty} F_n = (?)$ next page!

$\mathbb{R}$

$F_n = \left[\frac{1}{n}, 2\right] \quad (n \in \mathbb{N})$

$\Rightarrow \bigcup_{n\in\mathbb{N}} F_n = (0, 2]$

Proof

($\subset$) Let $x \in \bigcup_{n\in\mathbb{N}} F_n$.

i.e. $\exists n \in \mathbb{N} : x \in F_n$

i.e. $\exists n \in \mathbb{N}, \ \frac{1}{n} \leqq x \leqq 2$.

We show that $x \in (0, 2]$.

OK.

($\supset$) Let $x \in (0, 2]$.

i.e. $0 < x \leqq 2$.

We show that $x \in \bigcup_{n\in\mathbb{N}} F_n$.

i.e. $\exists n \in \mathbb{N}, \ \frac{1}{n} \leqq x \leqq 2$.

Take $n \in \mathbb{N}$ s.t. $\frac{1}{x} \leqq n$.

Then, $\frac{1}{n} \leqq x \leqq 2$.

//

$$\boxed{\begin{array}{l} \mathbb{R} \\ G_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \quad (n \in \mathbb{N}) \\ \Rightarrow \bigcap_{n \in \mathbb{N}} G_n = \{0\} \end{array}}$$

<u>Proof</u>

$(\supset)$ OK

$(\subset)$ Let $x \in \bigcap_{n \in \mathbb{N}} G_n$.

i.e. $\forall_{n \in \mathbb{N}}, \ x \in G_n$

i.e. $\forall_{n \in \mathbb{N}}, \ -\frac{1}{n} < x < \frac{1}{n}$ — $(*)$

We demonstrate that $\underline{x = 0}$.

Suppose by contradiction that $x \neq 0$.

Assume, w.l.g., that $x > 0$.

Choose $n \in \mathbb{N}$: $\frac{1}{x} \leq n$.

Then, $\frac{1}{n} \leq x$.

Thus, $\exists n \in \mathbb{N}$: $\frac{1}{n} \leq x$.

This contradicts $(*)$.

//

$$A_n = \left(0, \frac{1}{n}\right) \quad (n \in \mathbb{N})$$
$$\Rightarrow \bigcap_{n=1}^{\infty} A_n = \phi$$

Proof

Suppose by contradiction that

$$\exists x \in \bigcap_{n=1}^{\infty} A_n .$$

i.e. $\exists x \in \mathbb{R} : \forall n \in \mathbb{N},\ x \in A_n$

i.e. $\exists x \in \mathbb{R} : \forall n \in \mathbb{N},\ 0 < x < \frac{1}{n}$.

Choose $n \in \mathbb{N}$ such that $\frac{1}{x} \leq n$.

Then, $\frac{1}{n} \leq x$. for some $n \in \mathbb{N}$.

This is a contradiction.

//

Sets and set inclusion (1)

1. 集合Aを

$$A = \{\emptyset, a, \{b\}, \{c,d\}, \{e,b\}\}$$

と定める. 以下で正しいものはどれか？

(1) $a \in A$　(2) $a \subset A$　(3) $\{b,e\} \in A$　(4) $\{b,e\} \subset A$

(5) $c,d \in A$　(6) $\{e\} \in A$　(7) $\{\{b\},\{b,e\}\} \subset A$

(8) $\emptyset \in A$　(9) $\emptyset \subset A$

2. 次の集合のべき集合を答えよ.

(1) $A = \{a,b\}$　(2) $B = \{a,\{b\}\}$

3. 集合$X = \{a,b,c,d,e\}$を考え, 2^Xの2つの部分集合A, Bを

$$A = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, X\} \subset 2^X$$

$$B = \{\emptyset, \{a\}, \{a,c\}, X\} \subset 2^X$$

とする. (1) $A \cup B$ と (2) $A \cap B$ を答えよ.

4. 実数の集合$\mathbb{R}$の部分集合$[0,1)$を考える. 全体集合を下のように考えた場合, この集合の補集合はどうなるか？

(1) 全体集合を$\mathbb{R}$とした場合の補集合$[0,1)^C$

(2) 全体集合を$(-1,1)$とした場合の補集合$[0,1)^C$

5. 全体集合をX, Aをその部分集合とするとき, 次が正しいことを確認せよ.

(1) $X^C = \emptyset$　(2) $\emptyset^C = X$　(3) $(A^C)^C = A$

(4) $A \cup A^C = X$　(5) $A \cap A^C = \emptyset$

(6) $A \cup X = X$　(7) $A \cap X = A$

(8) $A \cup \emptyset = A$　(9) $A \cap \emptyset = \emptyset$

6. 全体集合をX, A, Bをその部分集合とする. このとき,

$$A \cap B = \emptyset \Leftrightarrow A \subset B^C$$

を証明せよ.

7. 実数の集合$\mathbb{R}$の部分集合族(部分集合の集まり)について, 以下を証明せよ.

(1) $F_n = [\frac{1}{n}, 2]$ $(n \in \mathbb{N})$ とするとき, $\bigcup_{n\in\mathbb{N}} F_n = (0,2]$.

(2) $G_n = (-\frac{1}{n}, \frac{1}{n})$ $(n \in \mathbb{N})$ とするとき, $\bigcap_{n\in\mathbb{N}} G_n = \{0\}$.

(3) $A_n = (0, \frac{1}{n})$ $(n \in \mathbb{N})$ とするとき, $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$.